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ABSTRACT

This guide represents the final experimental version of a pilot project which was conducted in the United States between 1973 and 1976. The ideas and the manner of presentation are based on the works of Georges and Frederique Papy. They are recognized for having introduced colored arrow drawings ("papygrams") and models of our numeration system (the Papy "minicomputer") into the teaching of mathematics at the elementary and secondary level in Belgium. This program follows the "spiral approach." In this guide, there is a review of some basic numerical activities which stem from the minicomputer. The standard minicomputer moves are reviewed. Several numerical games are given with the intention of providing a review of number concepts. (Author/SA)

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Comprehensive School Mathematics Program
CEMREL, Inc.

Experimental Version
4-[illegible]

CSMP Mathematics for the Intermediate Grades
Part II

Teacher's Guide

The World of Numbe

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
NATIONAL INSTITUTE OF EDUCATION

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Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades
Part II

The World of Numbers

CEMREL, Inc.

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SUGGESTED SCHEDULE FOR THE N – LESSONS

SUGGESTED SCHEDULE FOR THE N – LESSONS

WEEK	The World of Numbers	Materials
1	**1N** N1.1: Review: Minicomputer Golf p. 3	Teacher: Minicomputer set;* colored chalk Student: Pencil; paper; hand-calculator;** Worksheets N01, N02, N03
2	**3N** N2.1: Review of Multiplication p. 22	T: Colored chalk S: Pencil; paper; Worksheets N06, N07, N08
3	**5N** N3.1: Decimal Numbers and the Minicomputer p. 88	T: Minicomputer set; colored chalk; hand-calculator S: Pencil; paper; hand-calculator; Worksheets N31, N32, N33
4	**7N** N3.2: A Detective Story about Zip p. 95	T: Colored chalk; hand-calculator S: Pencil; paper; hand-calculator; Worksheets N34, N35
5	**9N** N3.3: Decimals and the Number Line #1 p. 100	T: Colored chalk; hand-calculator S: Pencil; paper; hand-calculator; Worksheets N36, N37, N38
6	**11N** N4.2: Telephone Game #2 p. 156	T: Colored chalk S: Pencil; paper; colored pencils; Worksheets N55, N56
7	**13N** N6.1: Subtraction #1 p. 182	T: Colored chalk S: Pencil; paper; Worksheets N60, N61, N62
8	**15N** N3.4: Decimals and the Number Line #2 p. 108	T: Colored chalk S: Pencil; paper; Worksheets N39, N40, N41
9	**16N** N5.2: Guess My Rule #2 p. 174	T: None S: Pencil; paper

* A teacher's Minicomputer set consists of four demonstration Minicomputer boards and a sufficient number of demonstration Minicomputer checkers. A student's Minicomputer set consists of two individual Minicomputer sheets (two boards per sheet) and a card of individual checkers.

SUGGESTED SCHEDULE FOR THE N - LESSONS

The World of Numbers	Materials
2N N1.2: Review: Detective Story p. 12	T: Colored chalk; Minicomputer set S: Pencil; paper; Worksheets N04, N05
4N N2.2: 10 × and ÷ 10 p. 28	T: Colored chalk; Minicomputer set S: Pencil; paper; Worksheets N09, N10, N11
6N N2.3: Composites of □ × and 10 × p. 34	T: Colored chalk S: Worksheets N12, N13, N14
8N N4.1: Telephone Game #1 p. 148	T: Colored chalk S: Colored pencils; pencil; paper; Worksheets N52, N53, N54
10N N5.1: Guess My Rule #1 p. 164	T: Hand-calculator; colored chalk S: Hand-calculator; Worksheets N57, N58, N59
12N N2.4: Multiplication Algorithm #1 p. 42	T: Colored chalk S: Pencil; paper; Worksheets N15, N16, N17
14N N7.1: The Relation, "Is A Positive Divisor Of" p. 200	T: Colored chalk S: Colored pencils; pencil; paper; Worksheets N66, N67, N68
ADJUSTMENT DAY	
17N N2.5: New Checkers on the Minicomputer p. 48	T: Minicomputer set; Multiplication Checker set S: Pencil; paper; Worksheets N18, N19, N20, N21

** For all lessons listing hand-calculators as materials, see the General Introduction Section 2(f) on Role and Use of Hand-Calculators. While two hand-calculators are adequate for these lessons, the ideal situation would be for each student to have one. Encourage students to bring their own hand-calculators.

SUGGESTED SCHEDULE FOR THE N - LESSONS

WEEK	The World of Numbers	Materials
10	**18N** N3.5: Newtonian Quiz #1 p. 118	T: Minicomputer set; colored chalk S: Pencil; paper; hand-calculator; Worksheets N42, N43, N44
11	**20N** N9.1: Fractions #1 p. 260	T: Colored chalk S: Pencil; paper; Worksheets N84, N85, N86
12	**22N** N10.1: Binary Abacus p. 292	T: Colored chalk S: Pencil; paper; Worksheets N94, N95, N96
13	**24N** N6.2: Subtraction #2 p. 189	T: Colored chalk S: Pencil; paper; Worksheets N63, N64, N65
14	**25N** N3.6: Multiples of Decimal Numbers p. 130	T: Colored chalk; hand-calculator S: Pencil; paper; hand-calculator; Worksheets N45, N46, N47
15	**27N** N7.3: Hand-Calculator Relations #2 p. 220	T: Colored chalk; hand-calculator S: Pencil; paper; hand-calculator; Worksheets N72, N73, N74
16	**29N** N3.7: Newtonian Quiz #2 p. 136	T: Colored chalk S: Pencil; paper; Worksheets N48, N49, N50, N51
17	**31N** N2.8: How Many Ways? p. 75	T: Minicomputer set; Multiplication Checker set; colored chalk S: Pencil; paper; Worksheets N28, N29, N30
18	**32N** N9.3: Fractions #3 p. 279	T: Colored chalk S: Pencil; paper; Worksheets N90, N91, N92, N93

SUGGESTED SCHEDULE FOR THE N - LESSONS

The World of Numbers	Materials
19N N8.1: Division #1 p. 244	T: Colored chalk S: Pencil; paper; colored pencils; Worksheets N79, N80
21N N7.2: Hand-Calculator Relations #1 p. 208	T: Hand-calculator; colored chalk S: Hand-calculator; Worksheets N69, N70, N71
23N N2.6: Multiplication Algorithm #2 p. 55	T: Minicomputer set; Multiplication Checker set S: Pencil; paper; Worksheets N22, N23, N24
ADJUSTMENT DAY	
26N N8.2: Division #2 p. 251	T: Colored chalk S: Pencil; paper; Worksheets N81, N82, N83
28N N2.7: Multiplication Algorithm #3 p. 64	T: Minicomputer set; Multiplication Checker set S: Pencil; paper; Worksheets N25, N26, N27
30N N9.2: Fractions #2 p. 268	T: Colored chalk S: Pencil; paper; Worksheets N87, N88, N89
ADJUSTMENT DAY	
33N N7.4: Hand-Calculator Relations #3 p. 229	T: Minicomputer set; Multiplication Checker set; hand-calculator; colored chalk S: Hand-calculator; pencil; paper; Worksheets N75, N76, N77, N78

SUGGESTED CHRONOLOGICAL ORDER OF N-LESSONS

Lesson One N1.1 - Review: Minicomputer Golf ... 3

Lesson Two N1.2 - Review: Detective Story 12

Lesson Three N2.1 - Review of Multiplication 22

Lesson Four N2.2 - 10 × and ÷ 10 28

Lesson Five N3.1 - Decimal Numbers and the Minicomputer 88

Lesson Six N2.3 - Composites of □ × and 10 × ... 34

Lesson Seven N3.2 - A Detective Story About Zip ... 95

Lesson Eight N4.1 - Telephone Game #1 148

Lesson Nine N3.3 - Decimals and the Number Line #1 ... 100

Lesson Ten N5.1 - Guess My Rule #1 164

Lesson Eleven N4.2 - Telephone Game #2 156

Lesson Twelve N2.4 - Multiplication Algorithm #1 ... 42

Lesson Thirteen N6.1 - Subtraction #1 182

Lesson Fourteen N7.1 - The Relation, "Is A Positive Divisor Of" 200

Lesson Fifteen N3.4 - Decimals and the Number Line #2 ... 108

Lesson Sixteen N5.2 - Guess My Rule #2 174

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Lesson Nineteen N8.1 - Division #1 244

Lesson Twenty N9.1 - Fractions #1 260

Lesson Twenty-one N7.2 - Hand-Calculator Relations #1 ... 208

Lesson Twenty-two N10.1 - Binary Abacus 292

Lesson Twenty-three N2.6 - Multiplication Algorithm #2 ... 55

Lesson Twenty-four N6.2 - Subtraction #2 189

Lesson Twenty-five N3.6 - Multiples of Decimal Numbers ... 130
Lesson Twenty-six N8.2 - Division #2 251
Lesson Twenty-seven N7.3 - Hand-Calculator Relations #2 ... 220
Lesson Twenty-eight N2.7 - Multiplication Algorithm #3 ... 64
Lesson Twenty-nine N3.7 - Newtonian Quiz #2 136
Lesson Thirty N9.2 - Fractions #2 268
Lesson Thirty-one N2.8 - How Many Ways? 75
Lesson Thirty-two N9.3 - Fractions #3 279
Lesson Thirty-three N7.4 - Hand-Calculator Relations #3 ... 229

THE WORLD OF NUMBERS

INTRODUCTION

By this time, your students have had a rich variety of experiences with various kinds of numbers and operations and relations on them. These include exposure to both positive and negative integers, decimal numbers, some rational numbers, numerical functions (e.g., $2\times$; $+3$; $\div 10$; -5), order relations (e.g., $<$ and $>$), and the notions of multiples and positive divisors of a given number. They have been introduced to paper and pencil algorithms for subtraction and addition of whole numbers and decimal numbers, and they have had extensive experiences using systematic methods for multiplication and division in preparation for algorithms.

In the present strand, these previous numerical experiences will be revisited, extended, and deepened. New situations will be presented, including the introduction of several fascinating extensions and modifications of numerical games: Minicomputer Golf (Section 1.1 and Section 3.5); Guess My Rule (Chapter 5); Telephone Game (Chapter 4); and Detective stories (Section 1.2 and Section 7.4); and new numerical games: Newtonian Quiz (Section 3.5 and Section 3.7) and Binary Abacus (Chapter 10). Old and new concepts will be explored in the context of these situations.

The first paragraph above, describing some of your students' previous numerical experiences, simply catalogues some of the realms of the world of numbers your students have visited. It does not indicate the travel routes nor the vehicles used to travel them.

The CSMP curriculum stresses the unity and continuity of the growth of mathematical ideas and concepts. This is reflected in numerous ways in this strand, as in the others. For example, CSMP employs the spiral approach, not pausing for mastery of each lesson but rather exposing the student to the elements of each content strand in different situations, interspersed with other strands throughout the year. It is important to recognize this approach consciously. If you strive for mastery of each single lesson, you will find yourself involved in a great deal of redundancy as the year progresses.

Further, CSMP presents the content in a situational framework. That is, a "pedagogy of situations" is employed which engages students in rich problem solving activities. These offer the opportunity both to develop necessary numerical skills and to gain deeper understanding of mathematical ideas and concepts in the world of numbers. At the same time the situations presented encourage the student to develop patterns of logical thinking and strategies for attacking problems.

Perhaps the most important embodiments of the CSMP approach are the three nonverbal languages used throughout its elementary program: the Papy Minicomputer, the language of arrows, and the language of strings. These are the vehicles which make it possible to develop the situations in which the content is presented to the student. Further, as the student grows in the CSMP program, these languages become basic investigatory tools available to explore the content.

It is hard to overstate the value of developing tools which aid in attacking problems as well as in representing situations; tools which are not confined to one area of mathematical content or to one level of the development of content. In the usual elementary curriculum, students must continually learn new language to express each new piece of mathematics that is studied. Armed with the universally applicable languages of the CSMP program, the students grow more and more familiar with the syntax of these languages and are freed to explore new content directly in terms of these same languages rather than being forced to think of each new mathematical idea as tied to a certain new language. This is definitely not to say that the CSMP students do not learn the usual descriptive language of mathematics. However, in the CSMP program the usual descriptive language is not the vehicle for learning, but only for describing the ideas.

For specific examples of the situations and the ways the CSMP languages are used in this strand, we refer you to the short introductory remarks preceding each chapter and, in particular, to the lessons themselves.

A WORD ABOUT STANDARD ALGORITHMS

We mentioned above that the CSMP program seeks to develop the necessary skills in the numerical domain as well as deeper understanding of the underlying mathematical ideas. We are fully in agreement with the thesis that along with the growth of understanding in the world of numbers, there must be a concommitant growth of familiarity and facility with numbers and the operations and relations on them. But facility must not be confused with understanding. They are partners in the growth of mathematical maturity, but it is extremely important that neither be sacrificed for the other. Rather, a balanced growth in each must be maintained.

In the usual elementary curriculum, paper and pencil algorithms for the four basic operations provide a prime opportunity for the abuse of the position stated above. It is undeniably important that the student eventually learns mechanical algorithms for the basic operations. However, if these algorithms are presented prematurely they stunt the student's ability to develop alternative algorithms, to do mental arithmetic, to estimate, and so on.

For example, rather than employing the standard algorithm to determine the standard name for 1,791 ÷ 9, it may be easier and more efficient to note that 200 x 9 = 1,800, so that 199 x 9 = 1,791 and 1,791 ÷ 9 = 199. Indeed, built into this way of approaching the problem is a relatively good estimate (1,800 ÷ 9) of the number involved.

If we insist on a mechanistic response to problems such as the one above (i.e., insist on the use of the paper and pencil algorithm), we may not only encourage inefficiency in many situations but also inhibit the development of the flexibility necessary for problem solving. On the other hand, if we present to students a rich array of situations in which they interact with numbers and their operations and let the standard algorithms grow naturally as just one facet of the students' experience, we obtain two important results. The students will gain the necessary facility with standard algorithmic procedures and at the same time retain the openness required to respond creatively to new situations in the world of numbers.

THE STANDARD MINICOMPUTER MOVES

Children who start the CSMP curriculum in Kindergarten or Grade One meet the Minicomputer for the first time just when they are starting to learn about numbers and the various operations on numbers. In such a context, the Minicomputer is used as a support for the positional system of numeration and as a calculating tool that helps to strengthen the first ideas of calculation and the learning of the "basic facts". To this end, heavy emphasis is laid on the "standard" representation of numbers on the Minicomputer (i.e., the representation that uses the fewest (positive) checkers possible digit by digit) and on the "standard" moves on the Minicomputer (these are described in detail below), because these provide the most efficient means to the end of teaching the children about elementary arithmetic.

Each of the "standard" Minicomputer moves leaves unchanged the number represented on the Minicomputer. Later, in the CSMP Upper Primary Grades Program (second and third grades), the children meet moves which either increase or decrease the number represented on the Minicomputer. Your students are already quite familiar with both the "standard" and "non-standard" Minicomputer moves. Many of them have probably developed shortcuts and may recognize the answer to a problem without putting it in "standard form".

Nonetheless, it is important for the students to realize that there is a mechanical method for performing all the "standard" moves. It requires no intuition and could be done by a suitably designed machine. For this reason the "standard" moves become known in the Intermediate Grades Program as "robot" moves. In cases where the mechanical nature of the Minicomputer is to be emphasized the students are asked to make "robot" moves and are limited to the following:

1 + 1 = 2 10 + 10 = 20 100 + 100 = 200 1,000 + 1,000 = 2,000	2 = 1 + 1 20 = 10 + 10 200 = 100 + 100 2,000 = 1,000 + 1,000
two checkers on the white square of a board = one checker on the red square of the same board	one checker on the red square of a board = two checkers on the white square of the same board

2 + 2 = 4 20 + 20 = 40 200 + 200 = 400 2,000 + 2,000 = 4,000	4 = 2 + 2 40 = 20 + 20 400 = 200 + 200 4,000 = 2,000 + 2,000
two checkers on the red square of a board = one checker on the purple square of the same board	one checker on the purple square of a board = two checkers on the red square of the same board

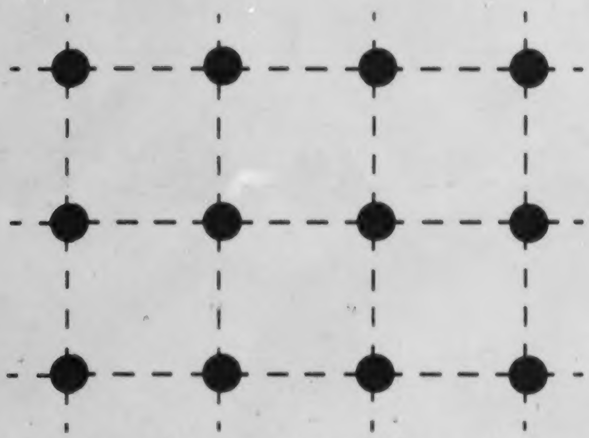
4 + 4 = 8 40 + 40 = 80 400 + 400 = 800 4,000 + 4,000 = 8,000	8 = 4 + 4 80 = 40 + 40 800 = 400 + 400 8,000 = 4,000 + 4,000
two checkers on the purple square of a board = one checker on the brown square of the same board	one checker on the brown square of a board = two checkers on the purple square of the same board

8 + 2 = 10 80 + 20 = 100 800 + 200 = 1,000 8,000 + 2,000 = 10,000	10 = 8 + 2 100 = 80 + 20 1,000 = 800 + 200 10,000 = 8,000 + 2,000
one checker on each of the brown and red squares of a board = one checker on the white square of the next board to the left	one checker on the white square of a board = one checker on each of the brown and red squares of the next board to the right*

* This provides the link that motivates the introduction of the decimal numbers. In particular, we have that one checker on the white square of the ones' board equals one checker on each of the brown and red squares of a special board (the tenths' board - or as it is often referred to in the lessons, the dimes' board) located to the right of the ones' board and separated from it by a special barrier.

NOTE ON THE ORDER OF MULTIPLICATION

Should the function "multiply by 2" be denoted by "2 ×" or "× 2"? Does it matter? The answer to this last question is "yes and no". It depends on your viewpoint and on the context in which you want to use it. In order to appreciate this fact a little better, let us go back to the very beginning: multiplication of nonnegative whole numbers. How does the following array of 12 dots appear to you?



If you see three rows of four (the red dashed lines), this fact can be recorded as:

$$\underset{\text{three rows}}{3} \quad \underset{\text{of}}{\times} \quad \underset{\text{four dots each}}{4} \quad \underset{\text{makes a total of}}{=} \quad \underset{\text{twelve dots}}{12}$$

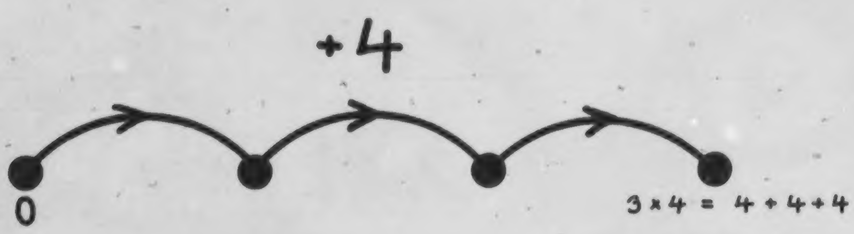
On the other hand, if you see four columns of three (the blue dashed lines), this fact can be recorded as:

$$\underset{\text{four columns}}{4} \quad \underset{\text{of}}{\times} \quad \underset{\text{three dots each}}{3} \quad \underset{\text{makes a total of}}{=} \quad \underset{\text{twelve dots}}{12}$$

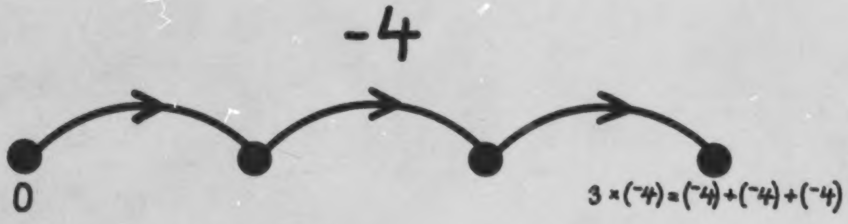
In that they describe two different perceptions, "4 × 3" and "3 × 4" are obviously very different. But, equally obvious is that "4 × 3" and "3 × 4" are both names for the same number, namely 12. So, in this case, it makes no difference if we denote "multiply by 3" by "×3" or "3 ×". This is true for any nonnegative integer, n: "×n" and "n×" may be used indifferently to denote "multiply by n", provided the number you are multiplying by n is also a nonnegative integer.

Problems do not arise until negative integers and (non-integer) decimal numbers put in their appearance; the reason being that it is a bit more difficult to find a model for multiplication when negative integers and (non-integer) decimal numbers are around. What would it mean, for example, to have an array of dots with six rows each containing negative two dots, or, even worse, negative three rows each containing negative five dots? What about 3.72 rows each containing seven dots? Fortunately, it is possible to find a model which allows the calculation of at least some products involving negative integers or (non-integer) decimal numbers; but, unfortunately, we lose the advantage of the array-of-dots model which makes it evident that the order of multiplication does not influence the answer. We call upon the language of arrows.

- 3 × 4 is the number obtained if you start from 0 and use three + 4 arrows.



- $3 \times ({}^{-}4)$ is the number obtained if you start from 0 and use three ${}^{-}4$ arrows.



This model has the advantage that it not only "pictures" the product, but it also provides a means of calculating that product: just label all the dots until you reach the one in which you are interested. The disadvantage is that it doesn't make any sense to speak of a negative number of arrows starting from 0. Thus, strictly speaking, your children have only been introduced to products in which the first number is a whole number (the second number can be either positive or negative or zero and can even be a (non-integer) decimal number). This has implications for notation.

- As noted above, if only nonnegative integers are involved, you can use "2 x" or "x 2" as the fancy takes you (we have scattered both of these notations throughout this semester's curriculum).

- If a negative integer (say, ${}^{-}4$) is being multiplied by 2 (say), then, strictly speaking, you should denote this "$2 \times ({}^{-}4)$" and not "${}^{-}4 \times 2$". This is because "$2 \times ({}^{-}4)$" carries the sense of "$({}^{-}4) + ({}^{-}4)$", whereas "${}^{-}4 \times 2$" has no natural connotation at this stage. In other words, strictly speaking, you should use "2 x" when negative integers and (non-integer) decimal numbers are involved, not "x 2". We have observed this rule in the Teacher's Guide for this semester.

We have emphasized the phrase "strictly speaking" above because this is not something you should make into a big issue with your children. In fact, it would be best to avoid all mention of this subject in your classroom if at all possible. If any of your children denote "times" functions one way when strictly speaking they should do it the other way, simply accept it without comment. There could be a variety of reasons why some of your children might do this; among them are:

(i) to conform to a pattern: all the other types of functions are usually written with the operation symbol on the left $(-2; \div 3; +6)$.

(ii) to generalize the results from the case of nonnegative integers: it is perfectly natural to believe that the order of multiplication makes no difference in the case of any integers.

Our concern in this note has been to make you aware of some of the underpinnings of this issue so that, in the very unlikely event that a problem arises in your class, you will have the necessary background information to understand why it might have occurred, thus perhaps pointing the way to a means of getting out of the difficulty.

Of course, you really have nothing to worry about; if your children infringe on any of the above "rules", what they do will not be wrong (because, after all, it really doesn't make any difference in which order you multiply two integers). It is just that, at this stage, what they do will be unfounded.

A WORD ON THE USE OF HAND-CALCULATORS*

The hand-calculator provides another powerful tool for the investigation of problems and situations in the world of numbers. Appropriately, this tool is a natural extension of the Papy Minicomputer, since its internal structure is an electronic simulation of the Minicomputer. This analogy need not be consciously acknowledged by the student but serves to indicate that the hand-calculator is indeed a natural tool in the CSMP program.

Two hand-calculators are recommended materials for Part II of the CSMP Intermediate Grades Program. They are not included in your complete classroom set; however, they may be purchased from CSMP as separate items. There are extensive opportunities for the use of hand-calculators in the lessons of Part II, especially in Chapter 7. In addition, the supplementary material for the slow learner, "Math Play Therapy", relies heavily on the use of hand-calculators.

We are aware that it is unrealistic, indeed premature at this time, to develop extensive classroom materials which require that every student in the class have a hand-calculator. We are firmly convinced through our own experiences with children using hand-calculators, that the hand-calculator is a potent pedagogical tool when used properly. The time is not far off when its presence in the classroom will be as commonplace as the pencil. For the present, CSMP will continue to investigate the use of the hand-calculator in the classroom, particularly with slow learners, and will develop situations designed for its use.

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* See General Introduction, Section 2 (f) for a more detailed discussion on hand-calculators.

1

REVIEW

In earlier grades of the CSMP curriculum, the Minicomputer is used as a support for the positional system of numeration; for calculations; for modeling the basic operations involving whole numbers, integers and decimal numbers; for number patterns; for numerical games; and for mental arithmetic. These uses of the Minicomputer are continued in this and other chapters of the World of Numbers strand.

The language of arrows and the language of strings have interacted with the Minicomputer throughout the curriculum to pose interesting situations and to aid in thinking about them and recording information. This is also continued in the World of Numbers strand.

In this first chapter some of the basic types of numerical activities that stem from the Minicomputer are quickly reviewed, including moves that transform a number and the recording of the moves in an arrow picture. The main activity used is the Minicomputer Golf Game.

Detective stories have proved to be a powerful motivational device for getting students to bring together many strategies they have learned in thinking about numbers and their properties. Chapter 1 reviews some of these strategies, including arrow and string pictures, in the context of a Detective Story.

CAPSULE LESSON SUMMARIES

Section 1.1: Put prescribed positive numbers on the Minicomputer and read numbers from the Minicomputer. Increase and decrease numbers on the Minicomputer specified amounts by moving checkers. Play the Minicomputer Golf Game. (Worksheets N01, N02, N03)

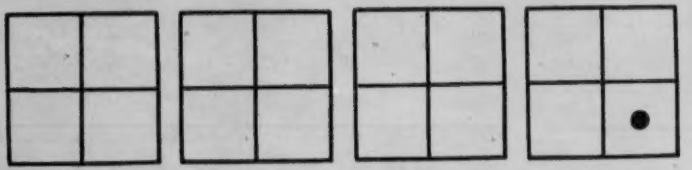
Section 1.2: Review positive prime numbers and positive divisors of specified numbers. Present a Detective Story whose clues involve string pictures, arrow pictures, and the Minicomputer. Present a partially labeled arrow picture and have students find ways to label the arrows. (Worksheets N04, N05)

1.1 Review: Minicomputer Golf

Exercise 1

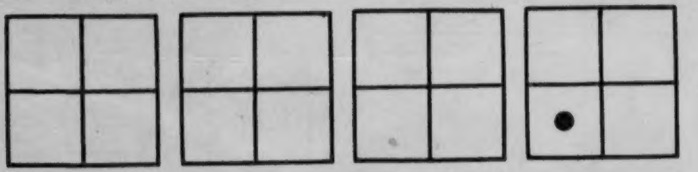
NOTE: After a three month vacation (i.e., for classes studying Part II during the first semester of the school year), students may need a quick review of the Minicomputer and this exercise provides that review. If it would be inappropriate for your class, feel free to omit the exercise.

Display four Minicomputer boards, put one checker on the 1-square, and take the students through a brief review in the following manner.



T: What number is this? (Answer: 1)

Ask the same question for each of these configurations.



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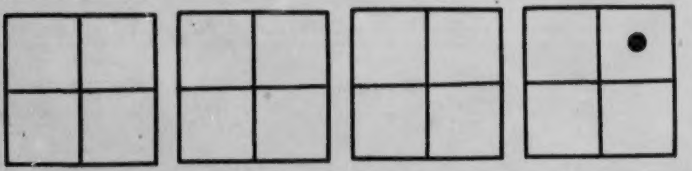

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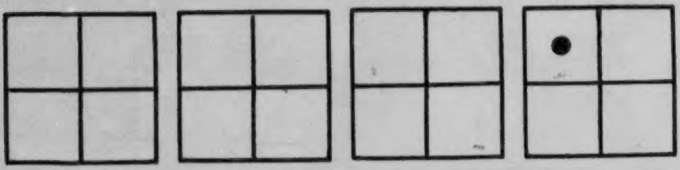

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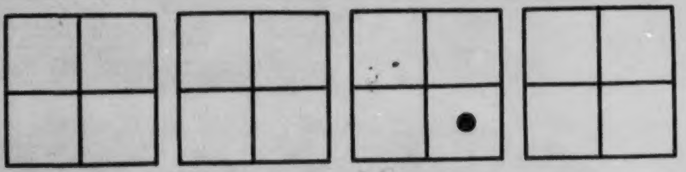
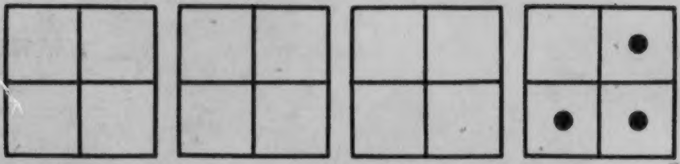


= 10

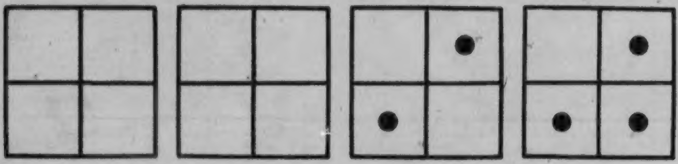
Continue with 20, 40, 80, 100, 200, 400, 800, and so on.

Clear the Minicomputer and put on 7.



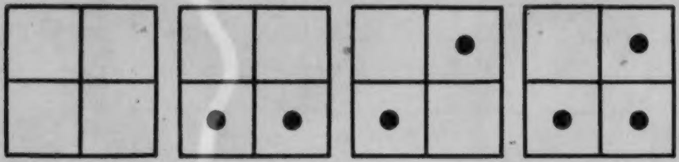
T: What number is this? (Answer: 7)

Put two checkers on the tens' board.



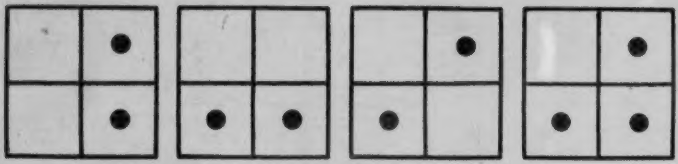
T: What number is this? (Answer: 67)

Put two checkers on the hundreds' board.



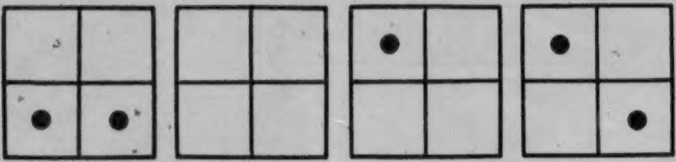
T: What number is this? (Answer: 367)

Put two checkers on the thousands' board.



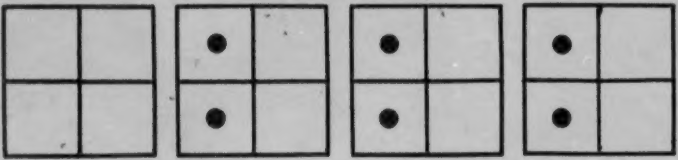
T: What number is this? (Answer: 5,367)

Repeat the activity using 9; 89; and 3,089.



= 3,089

Put this configuration on the board.



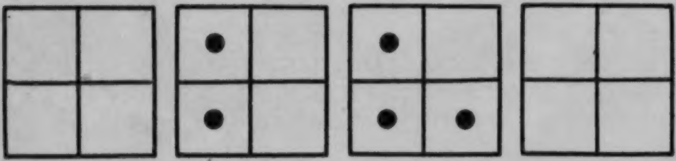
T: Write this number on your paper.

If some students have difficulty with this problem, ask:

T: Can you show the same number with one checker less?

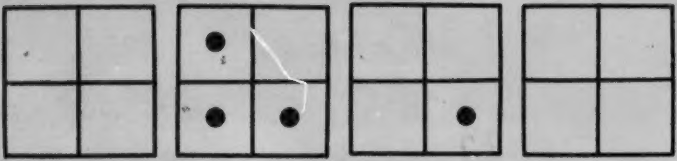
Let a student who is having difficulty make a move.

S: 8 + 2 = 10.



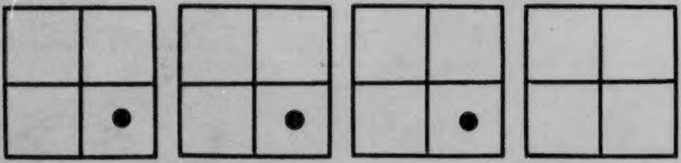
T: Can you show the same number again with one checker less?

S: 80 + 20 = 100.



T: Again, the same number with one checker less?

S: 800 + 200 = 1,000.

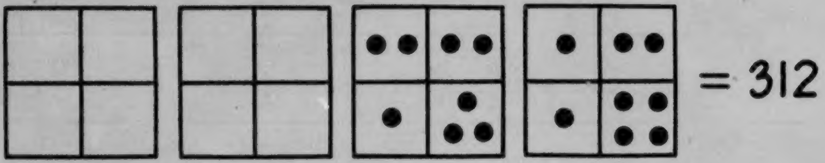


T: What is the number?

S: 1,110.

Exercise 2

Slowly and progressively put these checkers on the boards, a few at a time, and ask the students to calculate the number mentally.



T: Now I will do something to this number by moving or removing or adding some checkers. Each time, you should tell me if I get a number that is greater than, less than, or equal to the previous number.

Remove two checkers from the 40-square and put one checker on the 80-square.

T: What do you think?

S: It's the same.

T: Why?

S: Because 40 + 40 = 80.

Continue this activity making the following moves:

Move one checker from the 10-square to the 80-square.
(70 larger; 70 = 80 - 10)

Move one checker from the 20-square to the 2-square.
(18 smaller; 20 - 2 = 18)

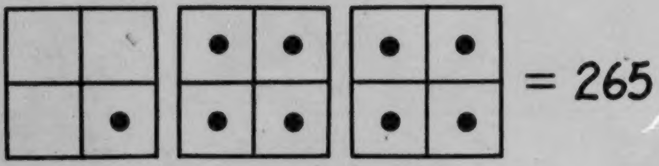
Move two checkers from the 80-square to the 40-square.
(80 smaller; 2 × 40 = 80)

Move four checkers from the 1-square to the 8-square.
(28 larger; 4 × 7 = 28)

Exercise 3

T: Now we will play a game of MINICOMPUTER GOLF.

Put these checkers on the Minicomputer and ask the students to calculate the number.



Divide your class into three groups. One group will be Team A, another group will be Team B, and the third group[*] (which may consist of only one or two students) will receive hand-calculators and will have to check progressively the results of the game.

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[*] If most students have calculators, this third group will not be necessary.

Record the goal and team colors on the chalkboard.

GOAL: 1,000

Team A: ——>——

Team B: ——>——

T: We are starting from 265 and the goal is 1,000. Students from Team A and from Team B will play in turns. Each time, they can move one checker from any square to any other square and thereby increase or decrease the number. The first team to reach the goal will be the winner.

Who would like to play first for Team A?

Let's suppose that the first player on Team A moves a checker from the 8-square to the 40-square. (We reproduce here the results of a game played in one of our development classes.)

T: Is the number on the Minicomputer larger or smaller?

S: Larger.

T: How much larger?

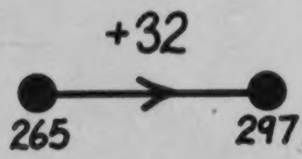
S: 32 larger.

T: What number is now on the Minicomputer?

S: 265 + 32 = 297.

Each time a play is made have the calculation checked by the third group with the hand-calculator.

Record Team A's play with this red arrow on the chalkboard.



Now a player from Team B takes a turn. The student moves a checker from the 20-square to the 80-square.

T: Is the number on the Minicomputer larger or smaller?

S: Larger.

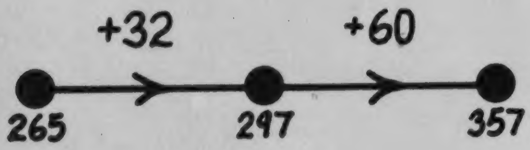
T: How much larger?

S: 60 larger.

T: What number is on the Minicomputer now?

S: 297 + 60 = 357.

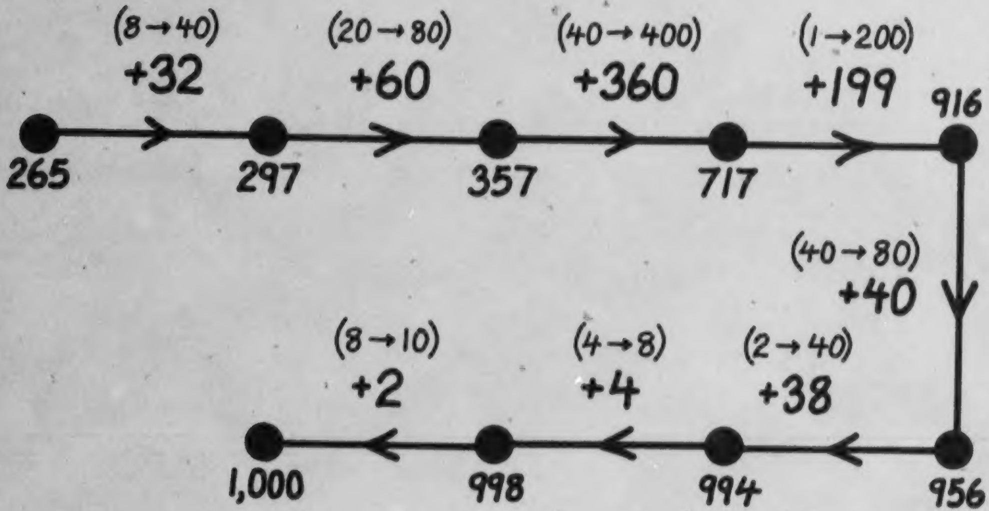
Record Team B's play with this blue arrow on the chalkboard.



Follow similar steps for the next players, alternately from Team A and Team B.

36

The results of our game are described by this red-blue road. Team B was the winner.


(8→40)
+32
(20→80)
+60
(40→400)
+360
(1→200)
+199
265
297
357
717
916
(40→80)
+40
(8→10)
+2
(4→8)
+4
(2→40)
+38
1,000
998
994
956

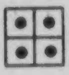
For a more detailed discussion of this game and other examples of actual play, see "Math Play Therapy" Volume I, Chapter 2 or the Teacher's Guide for CSMP Mathematics for the Intermediate Grades - Part I, "The World of Numbers". Sections M1.3, M3.2, and M3.3.

Worksheets N01, N02, and N03 are available.

Name

N 01 *

Put any number you wish on the Minicomputer using exactly four positive checkers.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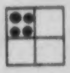
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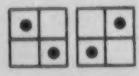
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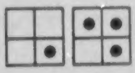
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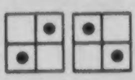
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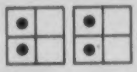
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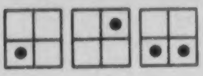
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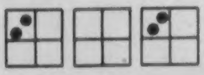
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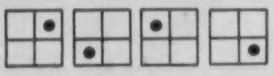
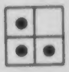

= 4281

Many solutions are possible.

Name

N 02 **

Put each of these numbers on the Minicomputer.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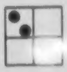
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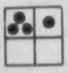
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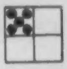
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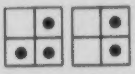
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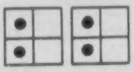
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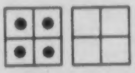
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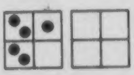
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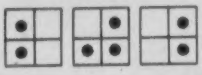
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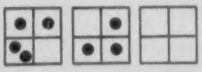
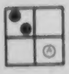

= 1,670

Many solutions are possible.

Name

N 03 ***

Write each of these numbers using exactly three checkers (positive or negative).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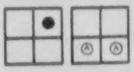
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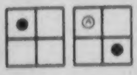
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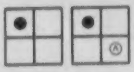
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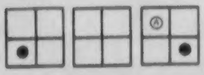
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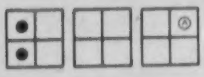
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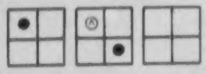
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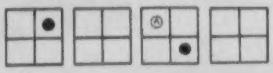

= 3,930

1.2 Review; Detective Story

Exercise 1

NOTE: After a three month vacation, students may need a quick review of positive divisors of a number. If it would be inappropriate for your class, feel free to omit Exercise 1.

T: Do you remember what the positive divisors of a number are? What are the positive divisors of 18?

S: 1, 2, 3, 6, 9, and 18.

If necessary, review the meaning of the divisors of a number. For example, 3 is a divisor of 18, because 18 is a multiple of 3, or because $18 \div 3 = 6$, or because $6 \times 3 = 18$.

Ask the same question concerning divisors of other numbers such as 27, 7, 36, 8, 15, etc.

T: Give me some numbers with exactly four positive divisors.

S: 6, because the positive divisors of 6 are 1, 2, 3, and 6.

S: 21; its positive divisors are 1, 3, 7, and 21.

Encourage many responses and for each one ask for class agreement.

T: Give me some numbers with exactly two positive divisors.

S: 7; its positive divisors are 1 and 7.

S: 3.

Other such numbers are: 2, 5, 11, 13, 17, 19, 23, 29, and so on.

T: What do we call the numbers with exactly two positive divisors?

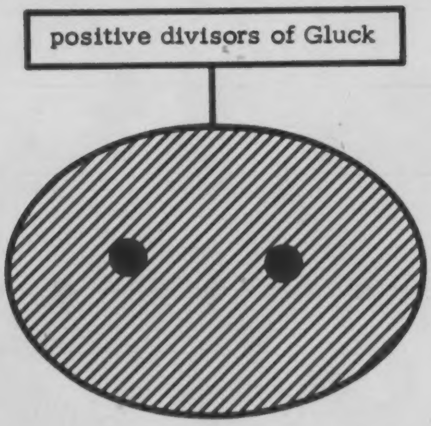
S: Prime numbers.

Exercise 2

Tell the class that they are going to solve a detective story about Gluck.

First clue

Draw this string picture on the board.



T: The hatching in this string picture means that Gluck has exactly two positive divisors. What are some numbers that Gluck could be?

S: 7.

S: 3.

S: 17.

S: Gluck could be any prime number.

Have students give more examples of numbers that Gluck could be. It would be helpful to make an ordered list of the numbers that the students suggest. Gluck can be any prime number.

T: Give me some numbers that Gluck cannot be.

S: 6.

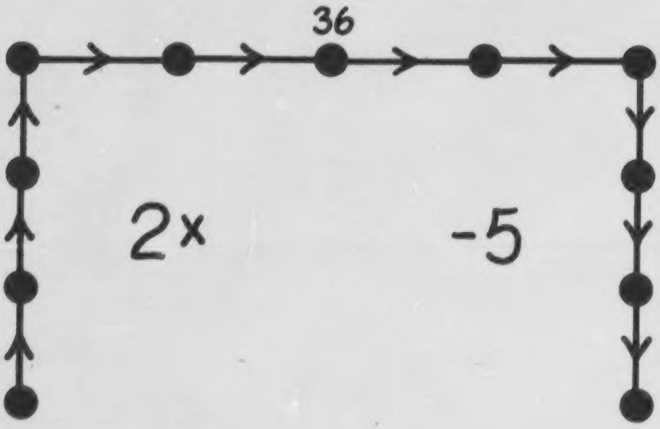
T: Why?

S: 6 has four divisors: 1, 2, 3, and 6.

Accept several examples from the students.

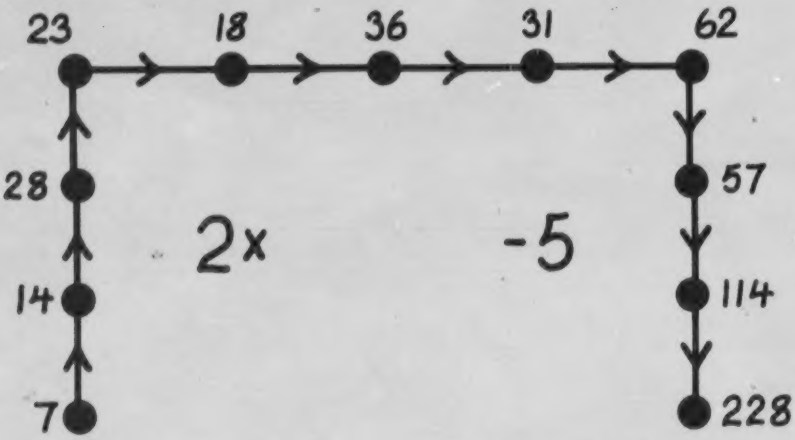
Second clue

Draw this arrow picture on the board.



T: Gluck is in this arrow picture.

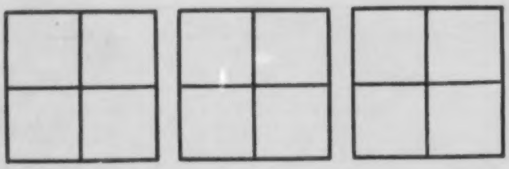
Have students label the dots in the arrow picture that is on the board. A completed arrow picture is shown below.



When all the dots have been labeled, the students should conclude that Gluck is 7, 23, or 31 since these are the only prime numbers in the arrow picture.

Third clue

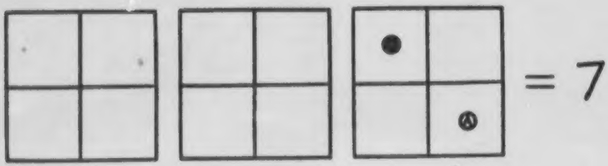
Display three Minicomputer boards.



T: Gluck cannot be put on the Minicomputer with exactly two checkers (positive or negative). Can Gluck be 7?

S: No, 7 can be put on the Minicomputer with exactly two checkers.

T: Show us.



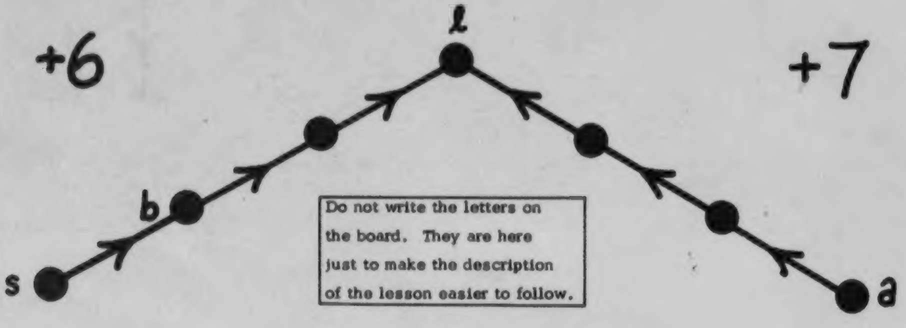
T: What about 23 and 31?

S: You must use at least three checkers for each.

The students should conclude that Gluck is 23 or 31.

Fourth clue

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

(The following questions are an introduction to the statement of the fourth clue.)

T: Where is the largest number in this arrow picture?

A student should point to the dot labeled "ℓ".

T: How much larger is this number (point to ℓ) than this one (point to b)?

S: 14.

T: How much larger is this number (point to ℓ) than this one (point to a)?

S: 19.

T: Good. And how much larger is this number (point to ℓ) than this one (point to s)?

S: 20.

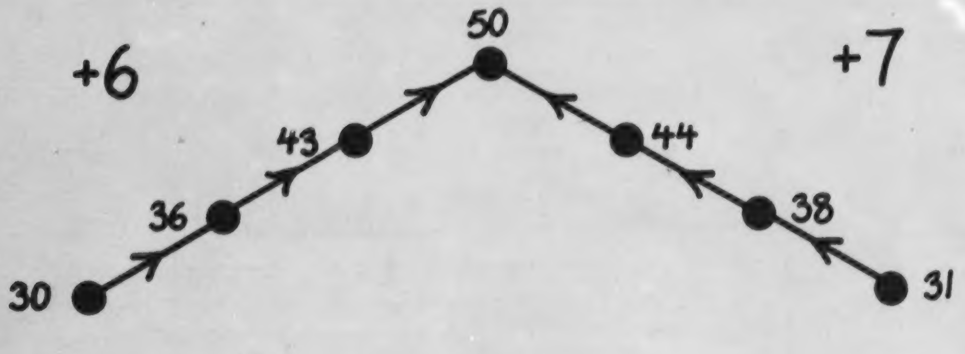
T: Where is the smallest number in our picture?

S (pointing to s): Here is the smallest.

T: Now, I will tell you the last clue.

The smallest number in this picture is 30, (label s "30") and Gluck is also in this arrow picture.

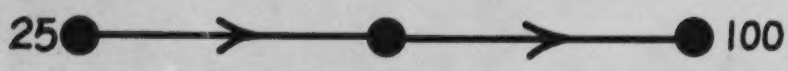
Have students label the other dots in the picture. A completed picture is shown below.



The students should conclude that Gluck is 31.

Exercise 3

Draw this arrow picture on the board.



T: What could the blue arrow be for?

Since this is an open activity, it is likely that your students' responses will be different from the responses in the dialogue below. Therefore, it will be necessary to adjust your reactions accordingly.

S: - 5.

T: If the blue arrow were for - 5, what could the red arrow be for?

S: 5 ×, because the middle dot would be for 20 and 5 × 20 = 100.

S: It could also be for + 80.

Continue this line of questioning and obtain several more solutions to the arrow picture. The table below shows some responses that students might make.

→	→
3 ×	+25
÷5	×20 or +95
+24	+51

Then reverse the situation, asking:

T: What could the red arrow be for?

Suppose a student suggests:

S: 2×.

T: If the red arrow were for 2×, then what could the blue arrow be for?

If a suggestion for the blue arrow is not forthcoming, ask:

T: - What number goes here (point to the middle dot)? Why?

S: 50, because 2 × 50 = 100.

Label the dot "50".

T: Now what could the blue arrow be for?

S: 2×.

S: +25.

Continue this activity as long as there is student interest. The table below shows some responses that students might make.

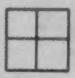
→	→
+175 or 8×	÷2
+125 or 6×	−50
+55	+20

Worksheets N04 and N05 are available for the rest of the class period.

Name ____________ N 04 *

First clue

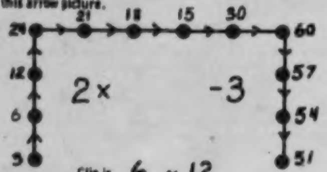
Clip is a secret number larger than 5 and can be put on this Minicomputer board using exactly two positive checkers.



Clip is 16 or 12 or 10 or 9 or 8 or 6.

Second clue

Clip is in this arrow picture.


24
21
18
15
30
60
12
6
3
2×
-3
57
54
51

Clip is 6 or 12.

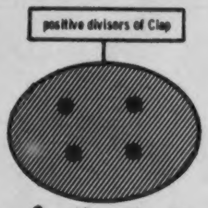
Third clue

Clip has exactly four positive divisors.

Clip is 6.

Name ____________ N 05 **

First clue

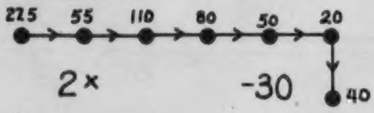

positive divisers of Clap

Clap could be 6 ; 8 ; 10 ; 14 ; 21 ; 27 ; 35 ;
55 ; and so on.

(Any number which is the product of two different prime numbers or which is the third power of a prime is a solution.)

Second clue

Clap is in this arrow picture.

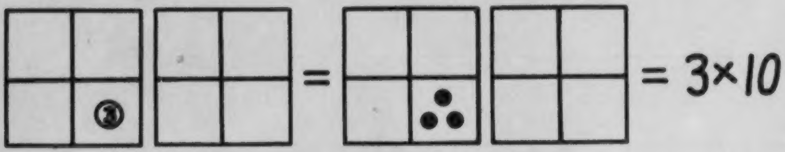

225
55
110
80
50
20
2×
-30
40

Clap is 55.

2

MULTIPLICATION

This chapter provides a broad range of situations involving multiplication and division, in which arrow diagrams and the Minicomputer are the principal supports. The arrow diagrams support the concept of, for example, 30 × as the composite of 3 × and 10 ×. A new kind of checker that is weighted by a number is used on the Minicomputer to achieve the same sort of results; for example,



The weighted checkers on the Minicomputer extend the range of mental arithmetic and types of calculations, which in this chapter extend to calculations such as "34 × 47" and to a form of the standard multiplication algorithm.

CAPSULE LESSON SUMMARIES

Section 2.1:	Review multiplication and division by identifying the arrows in a diagram that are labeled "×☐", "÷☐". Review positive multiples of numbers and common multiples in the format of string pictures. (Worksheets N06, N07, N08)
Section 2.2:	Using the Minicomputer and arrows for × 10, ÷ 10, set up experiences in multiplying numbers (including decimal numbers) by 10 and dividing numbers by 10. (Worksheets N09, N10, N11)
Section 2.3:	Multiply or divide numbers (including decimal numbers) by 20, by 30, etc., treating 20 × as the composite of 2 × and 10 ×, etc. (Worksheets N12, N13, N14)
Section 2.4:	Through composites, perform calculations such as "4 × 45", "40 × 45", "42 × 45", etc. (Worksheets N15, N16, N17)
Section 2.5:	Introduce weighted checkers (checkers with numerals printed on them) to denote the number of checkers on a square of the Minicomputer, and with this device perform calculations such as 5 × 484 = (5 × 400) + (5 × 80) + (5 × 4). (Worksheets N18, N19, N20, N21)
Section 2.6:	With the weighted checkers, reinforce calculations such as "30 × 42 = 3 × 420", and apply these to calculations such as "36 × 42". (Worksheets N22, N23, N24)
Section 2.7:	Engage in the activities of changing numbers on the Minicomputer by moving weighted checkers, and forming numbers using certain weighted checkers, again leading to calculations such as "34 × 47". (Worksheets N25, N26, N27)
Section 2.8:	Estimate a number written on the Minicomputer with many weighted checkers, and determine how it changes when one weighted checker is moved or traded for another. Determine the number of ways (via one-way roads) to go between certain cities visiting intermediate cities. Begin to recognize combinatorial strategies for solving such problems. (Worksheets N28, N29, N30)

2.1 Review of Multiplication

Exercise 1

This exercise is a brief warm-up activity using basic multiplication facts. Below are some facts that you could use. Feel free to adjust the exercise to the abilities of your class. A little later this semester we shall introduce a new way of looking at the basic multiplication facts, making use of the Minicomputer and a set of "new" checkers.

There are several methods one could use for this kind of warm-up activity. One teacher uses a soft, spongy ball about the size of a grapefruit. The teacher asks for a fact and throws the ball to a student. The student who catches the ball gives the answer and throws the ball back to the teacher. In this way the teacher maintains a quick and lively review.

2 × 5; 2 × 7; 2 × 9; 2 × 8;

3 × 5; 3 × 7; 3 × 9; 3 × 8.

$3 \times \square = 18$; $3 \times \square = 12$; $3 \times \square = 24$;

$5 \times \square = 25$; $5 \times \square = 40$; $8 \times \square = 32$;

$7 \times \square = 21$; $7 \times \square = 42$; $7 \times \square = 56$.

10 × 3 ; 10 × 7 ; 10 × 10;

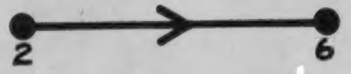
10 × 30 ; 10 × 24 ; 10 × 37;

10 × 12 ; 10 × 102 ; 10 × 125;

10 × 1,200; 10 × 201 ; 10 × 217.

Exercise 2

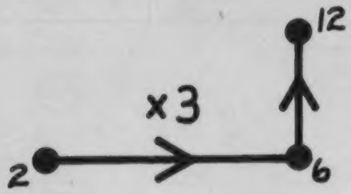
Draw this picture on the board.



T: How could we label this arrow "x□" or "+□" ? (Read "□" as "some number".)

S: x 3.

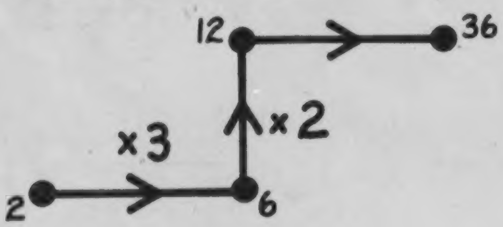
Label the arrow "x 3", and add this arrow to your picture.



T: How could we label this arrow, again "x□" or "+□"?

S: x 2.

Label the arrow "x 2", and extend the picture.

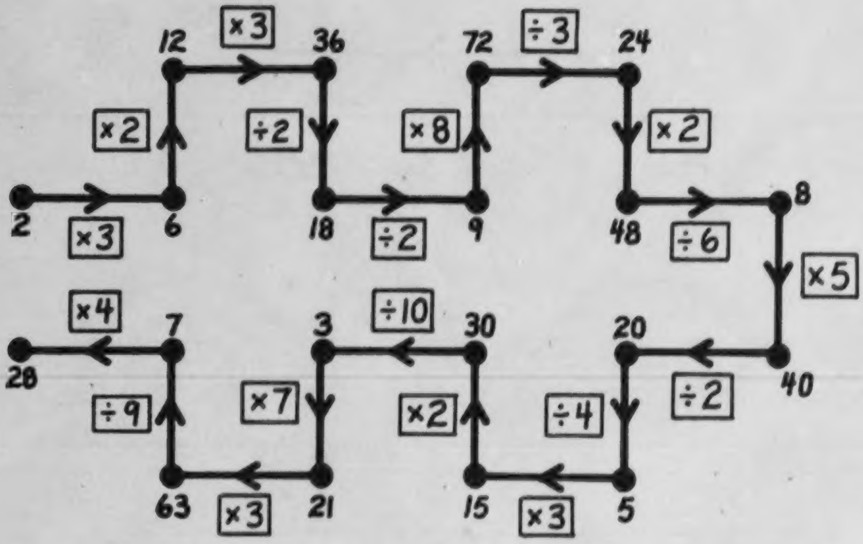


T: What could this arrow be for? Remember to use only "x□" or "+□".

S: x 3.

3N

Continue extending your picture. Below is a picture used in one of our development classes. Student responses are in the boxes.


12
×3
36
72
÷3
24
×2
÷2
×8
×2
2
×3
6
18
÷2
9
48
÷6
8
×5
×4
7
3
÷10
30
20
28
40
÷9
×7
×2
÷4
÷2
63
×3
21
15
×3
5

T: Can someone show us where we could draw × 6 arrows?

Have students come to the board and trace the × 6 arrows. In our picture, × 6 arrows can be drawn:

from 2 to 12;
from 6 to 36;
from 8 to 48;
from 5 to 30;
from 3 to 18;
from 12 to 72.

T: Can someone show us where we could draw more ÷ 4 arrows?

In our picture, + 4 arrows can be drawn:

from 36 to 9;

from 28 to 7;

from 12 to 3;

from 24 to 6;

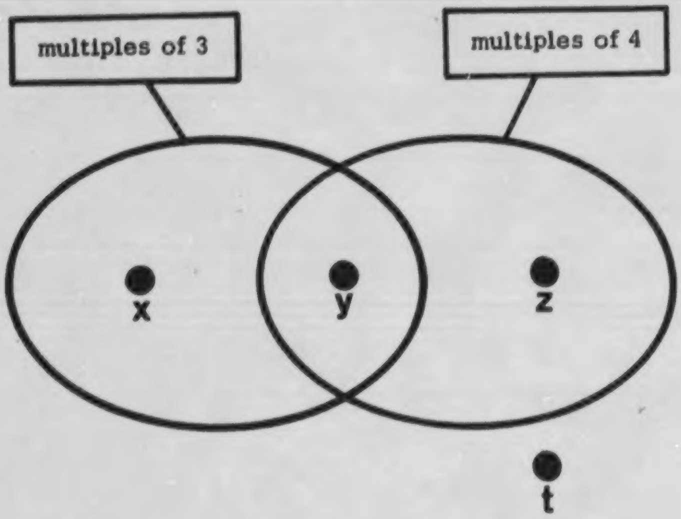
from 8 to 2;

from 72 to 18;

from 48 to 12.

Exercise 3

Draw this string picture on the board.


multiples of 3
multiples of 4
x
y
z
t

T: What are some numbers that x could be?

S: 3.

S: 6.

S: ⁻9.

S: 12.

T: What about 12?

S: It's a multiple of 3.

S: But it's also a multiple of 4. y could be 12.

T: What are some numbers that z could be?

Accept as correct any integer that is a multiple of 4 and not a multiple of 3; for example, $^{-}8$, $^{-}4$, 4, 8, 16, 20, 28, 32, and so on.

T: What are some numbers that y could be?

S: 12.

S: $^{-}24$.

S: 36.

Accept any number that is a multiple of 3 and a multiple of 4; for example, 0, $^{-}12$, 12, 24, 36, 48, 60, 72, and so on.

T: What do we call the numbers that y could be?

S: Multiples of 12.

T: What are some numbers that t could be?

Accept anything that is neither a multiple of 3 nor a multiple of 4; for example, $^{-}7$, $^{-}1$, 1, 2, 5, 7, 10, 0.5, 3.14, $\frac{2}{3}$, and so on.

Worksheets N06, N07, and N08 are available for the rest of the class period.

Name ____________________

N 06 *

2 × 5 = 10

2 × 8 = 16

2 × 10 = 20

3 × 5 = 15

3 × 7 = 21

3 × 10 = 30

4 × 3 = 12

4 × 4 = 16

4 × 6 = 24

4 × 9 = 36

5 × 3 = 15

5 × 5 = 25

5 × 8 = 40

$2 \times \boxed{7} = 14$

$2 \times \boxed{9} = 18$

$3 \times \boxed{3} = 9$

$3 \times \boxed{6} = 18$

$4 \times \boxed{2} = 8$

$4 \times \boxed{5} = 20$

$4 \times \boxed{10} = 40$

$5 \times \boxed{2} = 10$

$5 \times \boxed{4} = 20$

$5 \times \boxed{6} = 30$

$6 \times \boxed{2} = 12$

$6 \times \boxed{4} = 24$

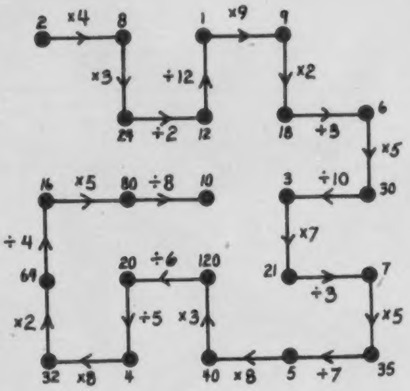
$6 \times \boxed{8} = 48$

$7 \times \boxed{3} = 21$

Name ____________________

N 07 **

Label each arrow "× ☐" or "÷ ☐".

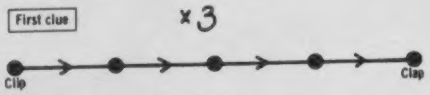

2
×4
8
×3
24
÷2
12
÷12
1
×9
9
×2
18
÷3
6
×5
30
÷10
3
×7
21
÷3
7
×5
35
÷7
5
×8
40
×3
120
÷6
20
÷5
4
×8
32
×2
64
÷4
16
×5
80
÷8
10

Name ____________________

N 08 ***

Clip and Clap are two secret numbers.

First clue



If Clip were 5, Clap would be 405.

If Clip were 7, Clap would be 567.

If Clap were 81, Clip would be 1.

Second clue

Clip is a positive divisor of 12.

Clip is 1 or 2 or 3 or 4 or 6 or 12.

Clap is 81 or 162 or 243 or 324 or 486 or 972.

Third clue

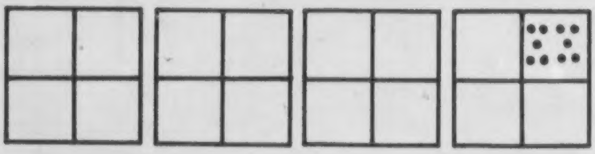
Clap can be put on the Minicomputer using exactly one positive checker and two negative checkers.

Clap is 972. Clip is 12.

2.2 10 × and ÷ 10

Exercise 1

Display four Minicomputer boards and put ten checkers on the 4 - square.



T: Can you suggest a calculation to find the number on the Minicomputer?

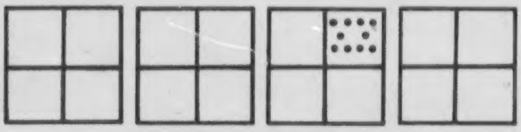
S: 10 × 4 = 40.

There are other correct answers, but this is the one you are most likely to get. Write this calculation on the chalkboard.

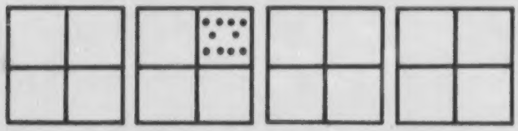
T: What is this number?

S: 40.

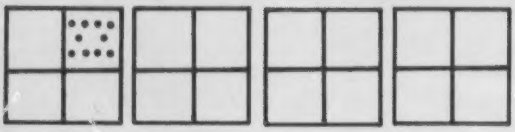
Ask the same questions for these configurations.



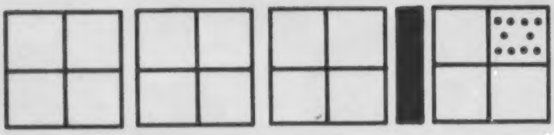
= 10 × 40 = 400



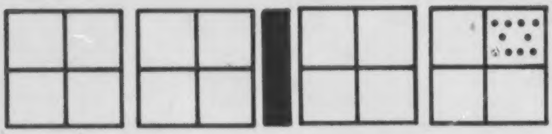
= 10 × 400 = 4,000



= 10 × 4,000 = 40,000



$= 10 \times 0.4 = 4$



$= 10 \times 0.04 = 0.4 = 0.40$

Exercise 2

Write these sets of problems on the board and let the class solve them.

(Answers are in the boxes.)

1) $10 \times 5 = \boxed{50}$

$10 \times 70 = \boxed{700}$

$10 \times 75 = \boxed{750}$

2) $10 \times 3 = \boxed{30}$

$10 \times 80 = \boxed{800}$

$10 \times 200 = \boxed{2{,}000}$

$10 \times 283 = \boxed{2{,}830}$

3) $10 \times 0.5 = \boxed{5}$

$10 \times 6 = \boxed{60}$

$10 \times 70 \quad \boxed{700}$

$10 \times 76.5 = \boxed{765}$

4) $10 \times 0.03 = \boxed{0.3}$

$10 \times 0.7 = \boxed{7}$

$10 \times 5 = \boxed{50}$

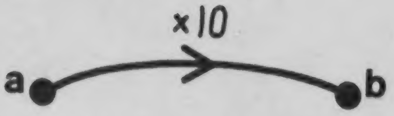
$10 \times 80 = \boxed{800}$

$10 \times 85.73 = \boxed{857.3}$

Exercise 3

Draw the following arrow picture.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



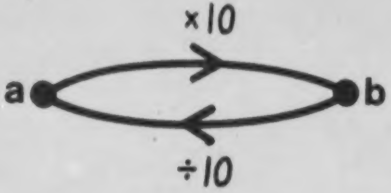
T: Let's play a game. I will label one dot then you must label the other. Suppose this dot (point to a) is 69 (label it "69"), then what could this dot be for (point to b)?

S: 690.

T: What could the return arrow be for?

S: ÷ 10.

Draw the return arrow and label it "÷ 10".



Continue with the exercise using the suggested values for a and b below. Student answers are in the boxes.

a	b
0.8	$\boxed{8}$
$\boxed{7}$	70
$\boxed{84}$	840
$\boxed{9.6}$	96

It is likely that the last problem will be difficult for some students.

T: Will this number (point to a) be larger or smaller than 96?

S: Smaller.

Most likely some student will suggest that the number is between 9 and 10. Be sure to try 9 and 10 to stress this. Then continue with these values for b. (Answers are in the boxes.)

a	b
$\boxed{0.3}$	3
$\boxed{0.09}$	0.9
$\boxed{0.45}$	4.5

If time permits, let a student come to the board and label one of the dots. Then the rest of the class can find the missing dot. This may be repeated as long as there is interest.

Distribute Worksheet N09, to the entire class. Worksheets N10 and N11 are available for students that complete N09.

After about 15 minutes when most students have finished Worksheet N09, put the drawings of N10 on the board. Let volunteers label the dots. The completed picture can be found at the end of this section.

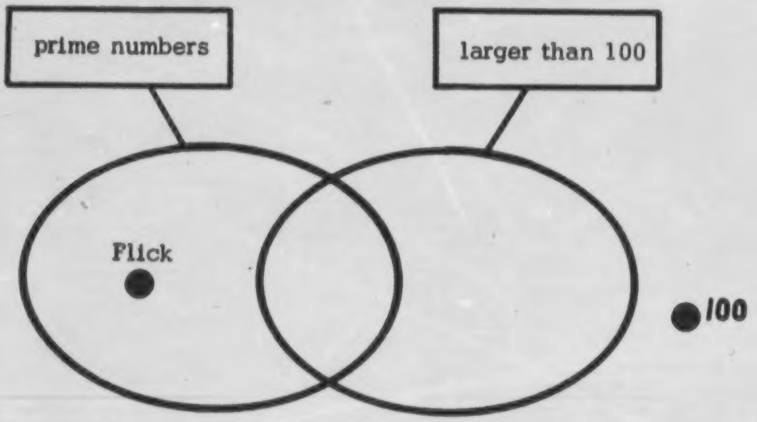
T: What does the string picture tell us about Flick? [Answer: Flick is a prime number and not larger than 100.]

It is possible that the location of 100 will be questioned.

T: Where in this string picture would I put 100?

4N

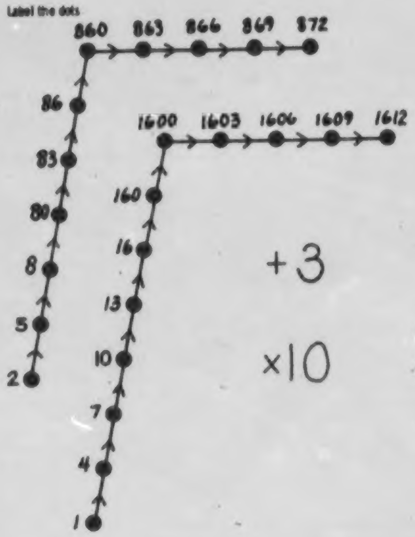
This question may elicit much discussion. Most students will agree 100 is not a prime but must give some thought to the "larger than 100" string. In a few minutes some students should suggest that 100 is not larger than 100. So, 100 would be outside of both strings as shown below.



From the arrow picture and the string picture, the class should eliminate all numbers in the arrow picture except 2 and 7. Then since Flick is odd, they should conclude Flick is 7.

Name ____________________ N 09 *

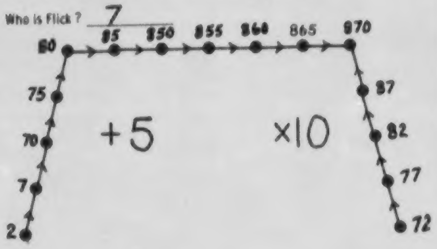
Label the dots

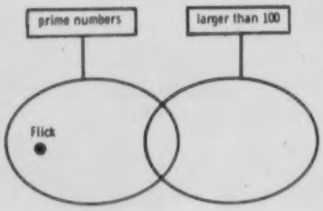

860
863
866
869
872
86
83
80
8
5
2
1600
1603
1606
1609
1612
160
16
13
10
7
4
1
+3
×10

Name ____________________ N 10 **

Flick is a secret odd number.
Flick is in this arrow picture and in this string picture.

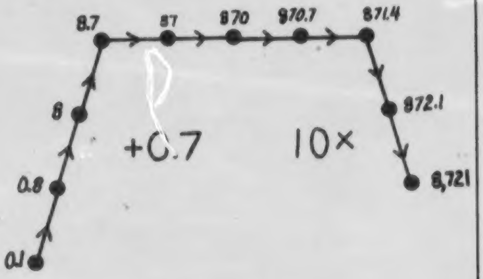
Who is Flick? 7


80
85
850
855
860
865
870
87
82
77
72
2
7
70
75
+5
×10

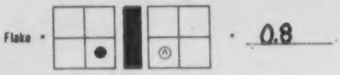

prime numbers
larger than 100
Flick

Name ____________________ N 11 ***

Flake is a secret number.
Flake is in this arrow picture.


8.7
87
870
870.7
871.4
872.1
8,721
8
0.8
0.1
+0.7
10×

Flake can be put on this Minicomputer using exactly one checker (positive or negative) on each board.

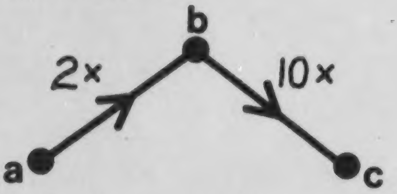


0.8

2.3 Composites of ☐ x and 10 x

Exercise 1

Draw this arrow picture on the board.



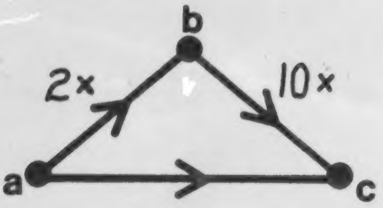
Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Suppose 9 is here (point to b and label it "9"). What number is here (point to c)? [Answer: 90]. What number is here (point to a)? [Answer: 4.5].

Repeat this situation several times. Below are some suggestions for you to consider. The boxes indicate student responses. Feel free to adjust the situations to the level of your class.

a	b	c
$\boxed{4.5}$	9	$\boxed{90}$
$\boxed{0.45}$	0.90	$\boxed{9}$
$\boxed{9}$	18	$\boxed{180}$
$\boxed{45}$	90	$\boxed{900}$

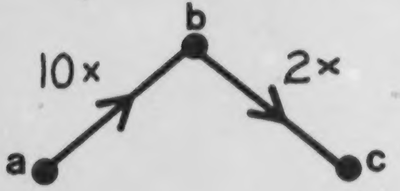
Add a green arrow to your picture.



T: What could the green arrow be for?

S: 20 ×.

Label the green arrow "20 ×". Now draw this arrow picture.



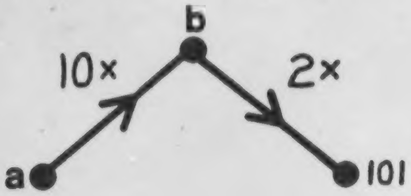
Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

Continue in a manner similar to the above.

a	b	c
$\boxed{24}$	$\boxed{240}$	480
$\boxed{0.16}$	1.6	$\boxed{3.20}$
$\boxed{0.7}$	7	$\boxed{14}$
$\boxed{5.05}$	$\boxed{50.5}$	101
$\boxed{4.3}$	$\boxed{43}$	86

Students enjoy trying to "stump the class". Let one or two students come to the board and label one dot while the rest of the class determines what the other two dots are.

Some students may have some difficulty with some of the problems. The following dialogue indicates how you might handle such a situation.



T: What number could go here (point to b)? Is the number larger or smaller than 101?

S: The number is smaller.

S: It's about 50.

T: What is one half of 100?

S: 50.

T: What is one half of 1?

S: 0.5 or $\frac{1}{2}$.

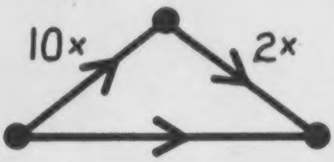
T: What is one half of 101?

S: 50.5.

T: What number goes here (point to a)?

S: 5.05.

Add a new arrow (drawn in black in the picture below) to your picture.



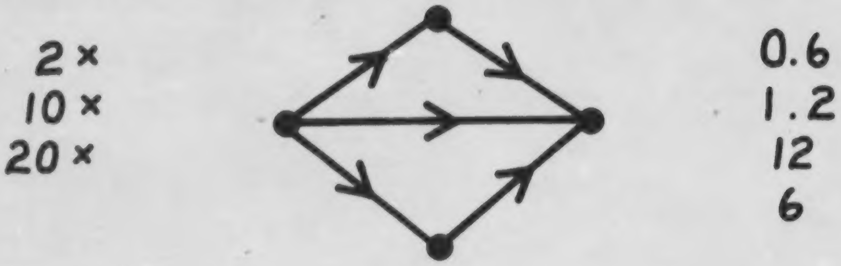
T: What could this new arrow be for?

S: 20×.

T: Yes, this new arrow could have been green.

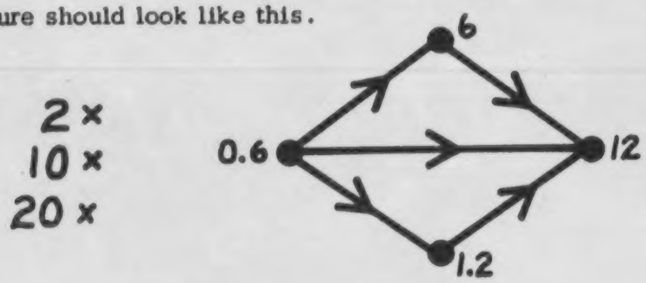
Replace the arrow with a green 20× arrow.

Draw this picture on the board.



T: I want you to label the dots using only the four numbers I have written to the side.

Let student volunteers come to the board and label the dots. The completed picture should look like this.

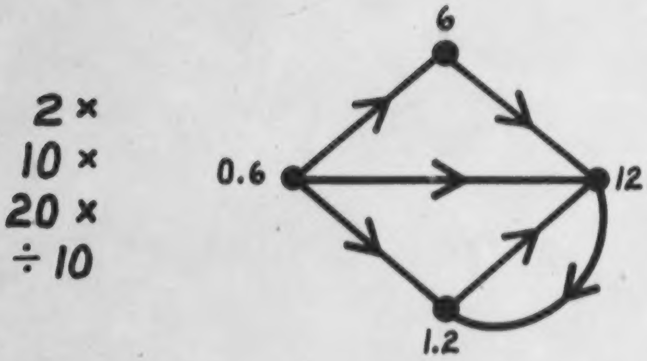


T: Use this picture to help you solve these problems.

Write these problems on the board.

$$12 \div 10 =$$
$$6 \div 10 =$$
$$12 \div 20 =$$

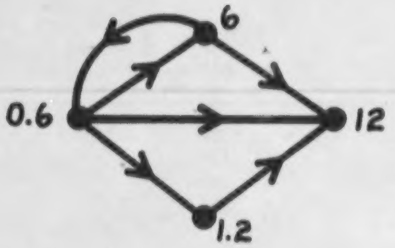
After a few minutes let some students come to the board, trace the return arrow, and give their solutions to the class.



S: 12 ÷ 10 = 1.2. (The student should trace the ÷ 10 return arrow.)

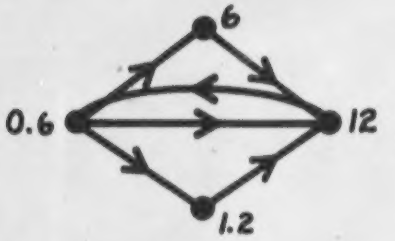
Below are solutions to the remaining two problems with the appropriate return arrow.

÷ 10



6 ÷ 10 = 0.6

÷ 20



12 ÷ 20 = 0.6

Exercise 2

Write these calculations on the board. (Answers are in boxes. Do not write them on the board.)

$$10 \times 23 = \boxed{230}$$

$$11 \times 23 = \boxed{253}$$

$$9 \times 23 = \boxed{207}$$

T: What number is 10 x 23?

S: 230.

T: What number is 11 x 23?

S: Just add one more 23. So the answer is 230 + 23 = 253.

T: What number is 9 x 23?

S: 207. Just subtract 23 from 230.

Write this set of problems on the board. (The answers are in boxes. Do not write them on the board.)

$$2 \times 17 = \boxed{34} \qquad 10 \times 17 = \boxed{170}$$

$$20 \times 17 = \boxed{340} \qquad 12 \times 17 = \boxed{204}$$

T: 2 x 17 = ?

S: 34.

T: Good. What number is 20 x 17?

S: 340; just multiply 34 by 10.

T: Very good. What number is 10 x 17?

S: 170.

T: Yes. What number is 12 x 17?

S: 204; add 34 and 170.

Do a similar exercise for this set of problems.

$3 \times 15 = \square$ $\qquad$ $10 \times 15 = \square$

$30 \times 15 = \square$ $\qquad$ $13 \times 15 = \square$

Exercise 3

Write these problems on the board.

$$15 \times 36 = 540$$
$$16 \times 36 = \square$$
$$14 \times 36 = \square$$

S: 16 x 36 = 576, because 540 + 36 = 576.

S: 14 x 36 = 504, because 540 - 36 = 504.

Use the following problems if time permits. (The answers are in the boxes. Do not write them on the board.)

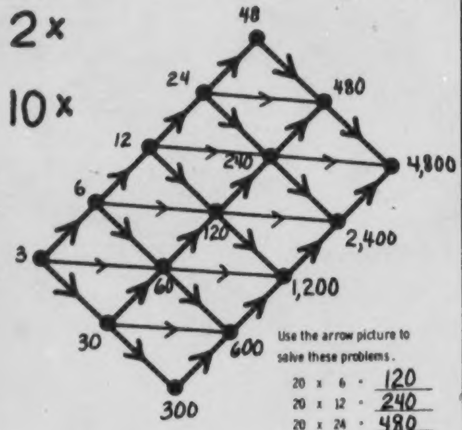
$$36 \times 75 = 2{,}700$$
$$37 \times 75 = \boxed{2{,}775}$$
$$35 \times 75 = \boxed{2{,}625}$$

$$45 \times 125 = 5{,}625$$
$$46 \times 125 = \boxed{5{,}750}$$
$$44 \times 125 = \boxed{5{,}500}$$

Worksheets N12, N13, and N14 are available.

Name ____________________ **N 12 ***

Label all the dots. Use green to draw as many 20x arrows as you can.

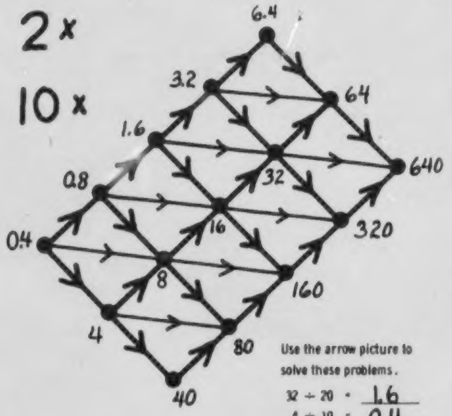

2x
10x
48
24
480
12
240
4,800
6
120
2,400
3
60
1,200
30
600
300

Use the arrow picture to solve these problems.

20 x 6 = 120
20 x 12 = 240
20 x 24 = 480
60 ÷ 10 = 6
120 ÷ 10 = 12
1,200 ÷ 10 = 120

Name ____________________ **N 13 ****

Label all the dots. Use green to draw as many 20x arrows as you can.

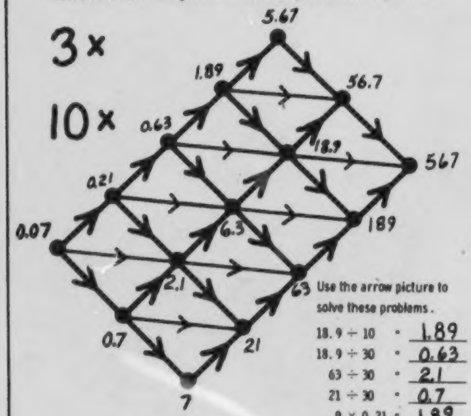

2x
10x
6.4
3.2
64
1.6
640
0.8
32
16
320
0.4
8
160
4
80
40

Use the arrow picture to solve these problems.

32 ÷ 20 = 1.6
4 ÷ 10 = 0.4
160 ÷ 20 = 8
8 ÷ 20 = 0.4
64 ÷ 20 = 3.2

Name ____________________ **N 14 *****

Label all the dots. Use green to draw as many 30x arrows as you can.


3x
10x
5.67
1.89
56.7
0.63
18.9
567
0.21
6.3
189
0.07
2.1
63
0.7
21
7

Use the arrow picture to solve these problems.

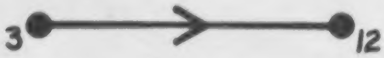
18.9 ÷ 10 = 1.89
18.9 ÷ 30 = 0.63
63 ÷ 30 = 2.1
21 ÷ 30 = 0.7
9 × 0.21 = 1.89
6.3 ÷ 9 = 0.7
0.63 ÷ 3 = 0.21
0.63 ÷ 9 = 0.07

2.4 Multiplication Algorithm #1

Exercise 1

T: I shall draw an arrow picture on the board. Each arrow should be labeled × some number or + some number.

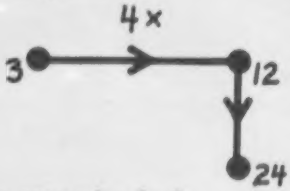
Draw this arrow picture on the board.



T: What could the blue arrow be for?

S: ×4.

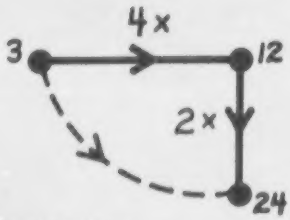
Label the arrow and extend the picture.



T: What could the blue arrow be for?

S: ×2.

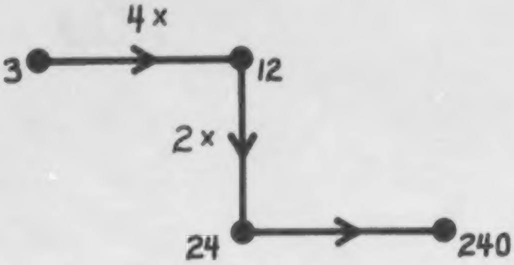
Add this new information to your picture. Trace with your finger the arrow that is shown as the dotted arrow in the picture below. (Don't draw the arrow.)



T: What could this arrow be for?

S: 8 x.

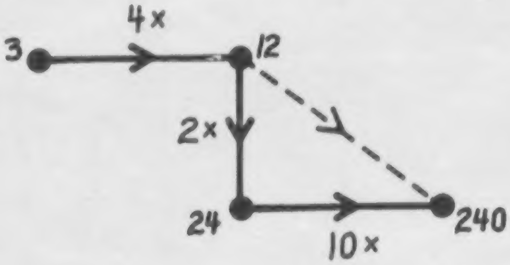
Extend the picture with an arrow from 24 to 240.



T: What could this arrow be for?

S: 10 x.

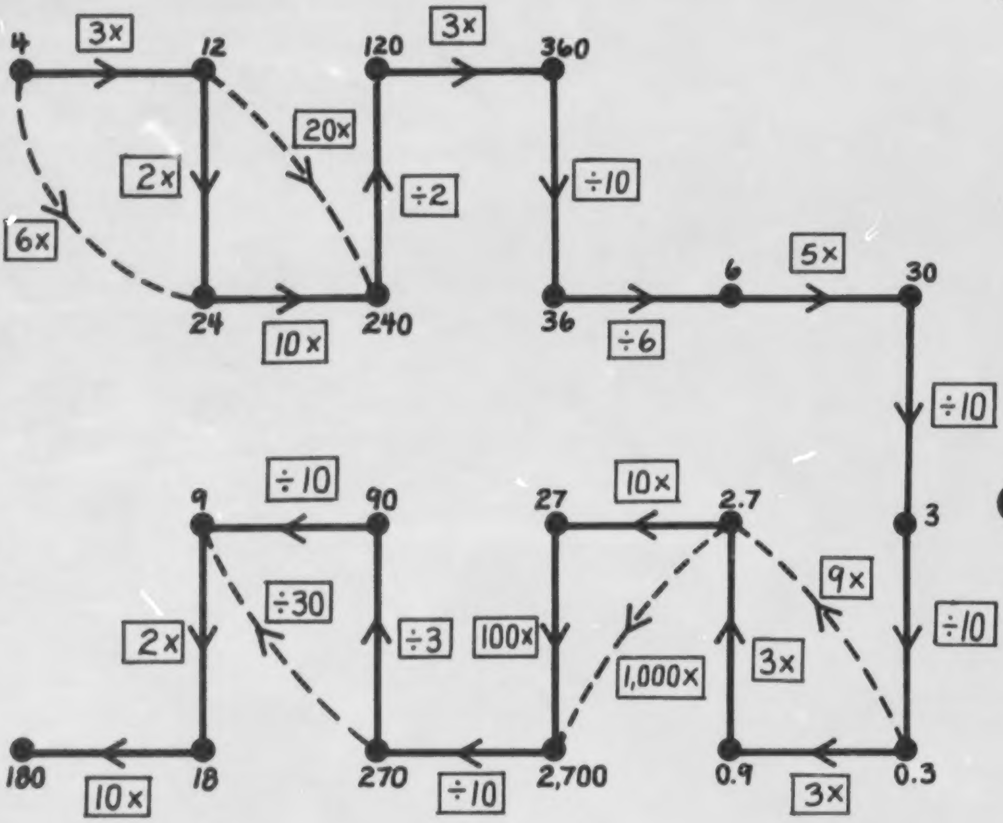
Label the arrow "10 x" and trace the arrow that is shown as the dotted arrow in the picture below. (Don't draw the arrow.)



T: What could this arrow be for?

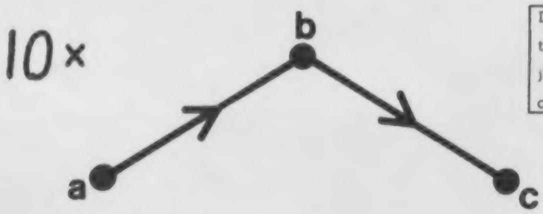
S: 20 x.

Continue extending the arrow picture as long as there is student interest. An arrow picture is drawn below for your use. A few dotted arrows have been included and correct responses have been boxed. Feel free to adjust the level of this exercise to your class. Be careful to pick numbers so that × or ÷ arrows are appropriate.


4
3×
12
120
3×
360
20×
2×
÷2
÷10
6×
6
5×
30
24
10×
240
36
÷6
÷10
9
÷10
90
27
10×
2.7
3
÷30
9×
2×
÷3
100×
÷10
1,000×
3×
180
10×
18
270
÷10
2,700
0.9
3×
0.3

Exercise 2

Draw this picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

Label the dot at b "75".

T: What is this number (point to c)?

S: 750, because 10 × 75 = 750.

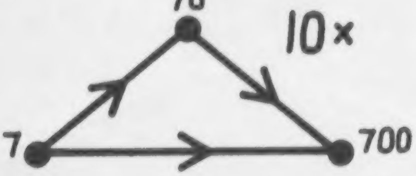
T: Good. What is this number (point to a)?

S: 7.5, because 10 × 7.5 = 75.

Repeat this situation twice more. Below are two suggestions for you to consider. The answers are in the boxes. Feel free to adjust the situations to the level of your class.

a	b	c
$\boxed{0.68}$	6.8	$\boxed{68}$
7	$\boxed{70}$	$\boxed{700}$

Add this blue arrow to your picture.



T: What could this blue arrow be for?

S: 20×.

T: Does 20 × 7 = 700?

S: No. It should be 100×, because 100 × 7 = 700.

Label the blue arrow "100×".

Exercise 3

Write this set of problems on the board. The answers are in the boxes.

$10 \times 45 = \boxed{450}$ $2 \times 45 = \boxed{90}$

$12 \times 45 = \boxed{540}$ $20 \times 45 = \boxed{900}$

T: I have written some problems on the board. Copy them on your paper and do the calculations. Try to do them in your head. When you have finished raise your hand.

Give the class a few minutes to work, them check their solutions collectively.

Encourage mental calculation such as

$10 \times 45 = 450$ and $2 \times 45 = 90$, so $12 \times 45 = 450 + 90 = 540$.

Leaving the worked problems on the board, continue the exercise with these problems.

$4 \times 45 = \boxed{180}$ $24 \times 45 = \boxed{1{,}080}$

$40 \times 45 = \boxed{1{,}800}$ $42 \times 45 = \boxed{1{,}890}$

Write this problem on the board and let a volunteer explain how he or she would work it.

$$\begin{array}{r} 72 \\ \times\ 12 \\ \hline \end{array}$$

Worksheets N15, N16, and N17 are available for the remainder of the lesson.

Name ____________ **N 15** *

$$\begin{array}{r} 375 \\ \times 2 \\ \hline 750 \end{array} \qquad \begin{array}{r} 2,344 \\ \times 3 \\ \hline 7,032 \end{array}$$

$$\begin{array}{r} 2,652 \\ \times 2 \\ \hline 5,304 \end{array} \qquad \begin{array}{r} 3,215 \\ \times 3 \\ \hline 9,645 \end{array}$$

10 x 25 · 250

2 x 25 · 50

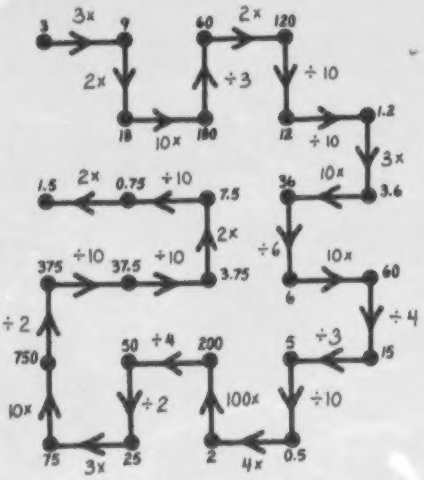
12 x 25 · 300

20 x 25 · 500

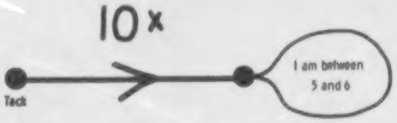
22 x 25 · 550

Name ____________ **N 16** **

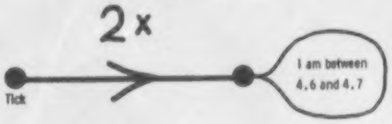
Label each arrow "☐ x" or " ÷ ☐ ".


3
3x
9
2x
18
10x
180
÷3
60
2x
120
÷10
12
÷10
1.2
3x
3.6
10x
36
÷6
6
10x
60
÷4
15
÷3
5
÷10
0.5
4x
2
100x
200
÷4
50
÷2
25
3x
75
10x
750
÷2
375
÷10
37.5
÷10
3.75
2x
7.5
÷10
0.75
2x
1.5

Name ____________ **N 17** ***


10x
Tack
I am between 5 and 6

Tack could be 0.51 ; 0.52 ; 0.53 ; 0.54 0.55 ; 0.56 ; 0.57 ; 0.58 ; 0.59 ; 0.583 ; 0.595 ; 0.555 ; and so on. (Any number between 0.5 and 0.6 is a solution.)


2x
Tick
I am between 4.6 and 4.7

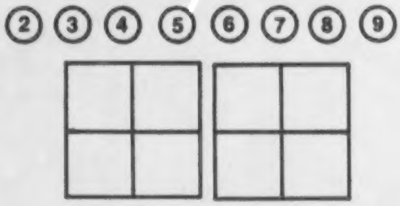
Tick could be 2.31 ; 2.315 ; 2.32 ; 2.324 ; 2.33 ; 2.337 ; 2.339 ; 2.3399 ; 2.34 ; 2.345 ; 2.3456 ; 2.34567 ; 2.349 ; 2.34999 and so on.

(Any number between 2.3 and 2.35 is a solution.)

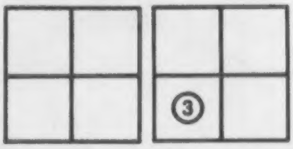
2.5 New Checkers on the Minicomputer

Exercise 1

Display two Minicomputer boards and your set of the new checkers.



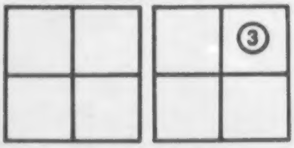
T: Today we will work with new checkers that have numbers on them. I am going to place a checker with a "3" [illegible] it on the Minicomputer. What is this number?



S: 6.

S: 3 x 2.

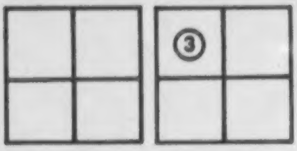
T: Good. What is this number?



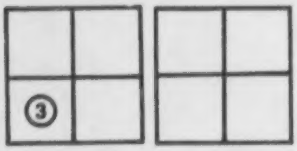
S: 12.

S: 3 x 4.

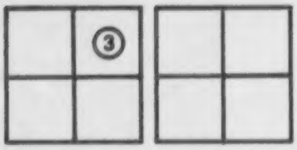
Similarly, move the ③ -checker to other squares.



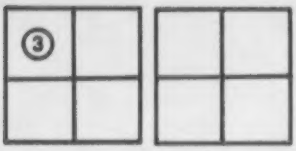
= $3 \times 8 = 24$



= $3 \times 20 = 60$



= $3 \times 40 = 120$

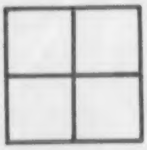


= $3 \times 80 = 240$

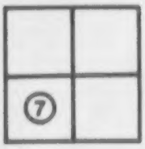
Continue in this manner using a ⑤ -checker and an ⑧ -checker.

Exercise 2

Display one Minicomputer board.

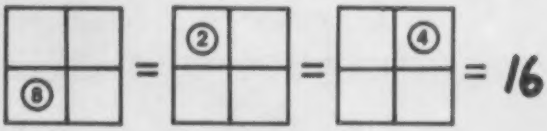


T: Who can show me 14 using exactly one of the new checkers?

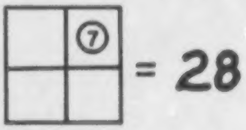


= 14

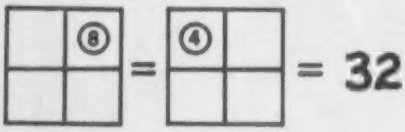
T: Who can show me 16 using exactly one of the new checkers?



T: 28 with exactly one of the new checkers?



T: 32 with exactly one of the new checkers?

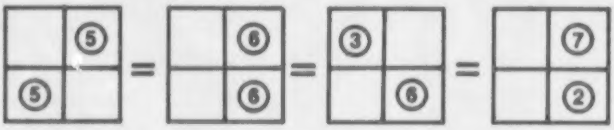


T: 30 with one of the new checkers?

Even though this is not possible using only one board, let several students try until most of the class is convinced that there is no solution.

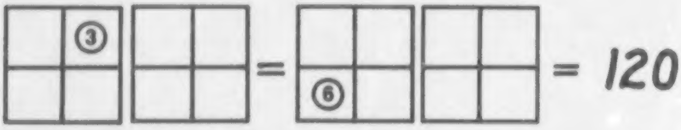
T: 30 with exactly two of the new checkers?

A few possibilities are

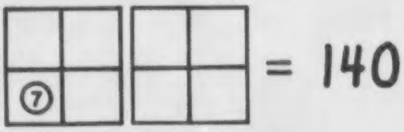


Add a ten's board and continue with the following situations.

T: 120 with exactly one checker?

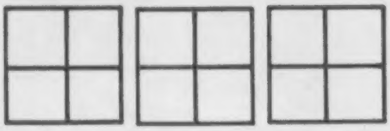


T: 140 with exactly one checker?

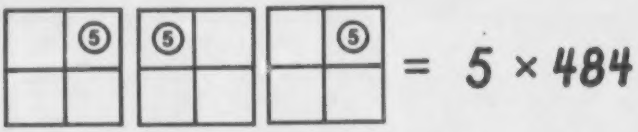


Exercise 3

Display three Minicomputer boards.



T: I would like someone to put 5 x 484 on the Minicomputer using the new checkers.



T: Let's use the Minicomputer to help us with the calculation.

Write the calculation on the chalkboard as the students say what is on each board.

S: 5 x 4 = 20 (on the ones' board).

S: 5 x 80 = 400 (on the tens' board).

S: 5 x 400 = 2,000 (on the hundreds' board).

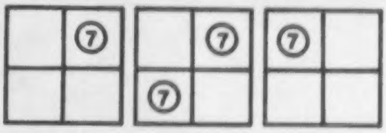
$$\begin{array}{rr} 5 \times 4 = & 20 \\ 5 \times 80 = & 400 \\ 5 \times 400 = & \underline{2{,}000} \\ & 2{,}420 \end{array}$$

S: So 5 x 484 = 2,420.

Write the problem on the board and let a student solve it in the usual manner.

$$\begin{array}{r} {}^{4\,2} \\ 484 \\ \times\ 5 \\ \hline 2,420 \end{array}$$

Repeat the exercise for "7 x 468".



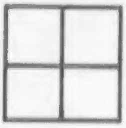
$= 7 \times 468 = 3,276$

Worksheets N18, N19, N20, and N21 are available for the remainder of the lesson.

It is possible that some students will finish before the end of class. In that case, you could write this detective story on the board.

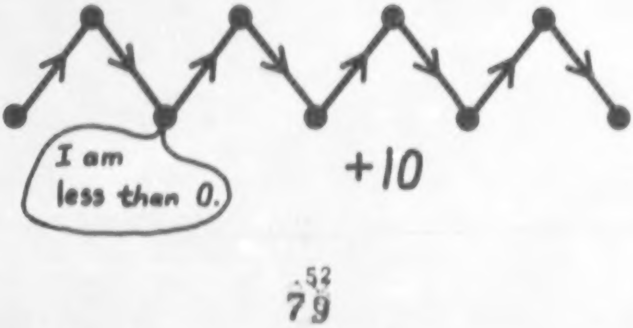
First Clue

Tok and Kot can be put on this Minicomputer board using exactly one of our new checkers.



Second Clue

Tok and Kot are in this arrow picture.



Third Clue

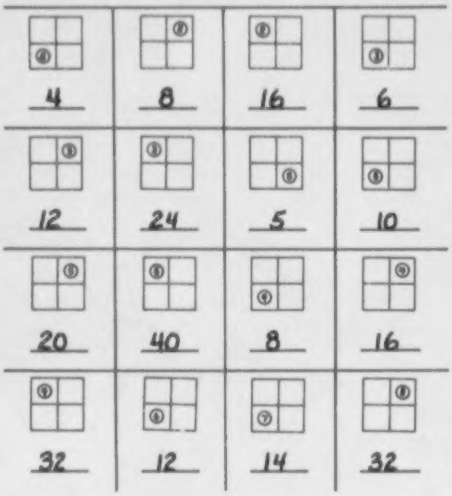
Tok and Kot are both square numbers.

Tok and Kot are ______________ and ______________.

The students should conclude that Tok and Kot are 16 and 36, respectively.

Name ____________________ **N 18** *

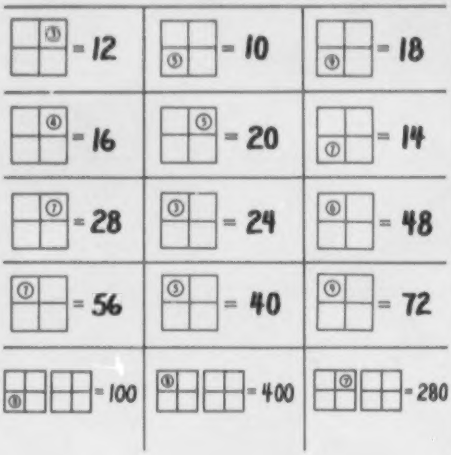
What number is written on the Minicomputer ?


4
8
16
6
12
24
5
10
20
40
8
16
32
12
14
32

Name ____________________ **N 19** **

Put these numbers on the Minicomputer using exactly one of these checkers :

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨


= 12
= 10
= 18
= 16
= 20
= 14
= 28
= 24
= 48
= 56
= 40
= 72
= 100
= 400
= 280

Other solutions are possible.

Name ____________________ **N 20** ***

Diz is a secret number.

First clue

Diz is a multiple of 6 and can be put on this Minicomputer using exactly one of these checkers :

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

Diz is 6 or 12 or 18 or 24 or 36 or 48 or 72 .

Second clue

If you add 1 to Diz, you will get a square number.

Diz is 24 or 48 .

Third clue

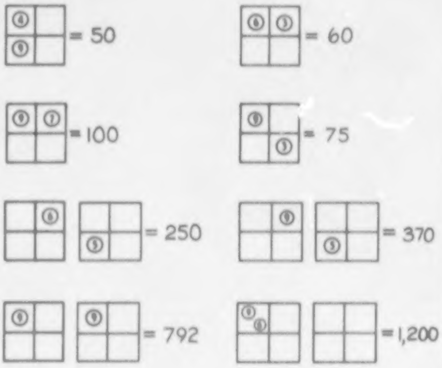
Diz has exactly ten positive divisors.

Diz is 48 .

Name ____________________ **N 21** ****

Put each of these numbers on the Minicomputer using exactly two of these checkers :

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

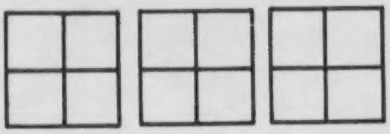

= 50
= 60
= 100
= 75
= 250
= 370
= 792
= 1,200

Other solutions are possible.

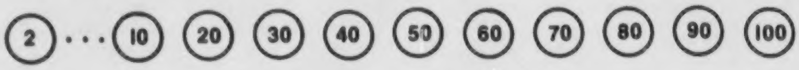
2.6 Multiplication Algorithm #2

Exercise 1

Display three Minicomputer boards

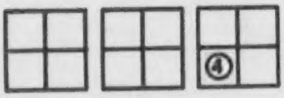


and your set of these checkers:

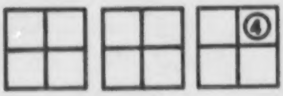


T: We will use these checkers today. I will put a number on the Minicomputer and you must tell me what it is.

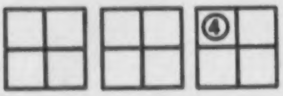
Have the class decode the following:



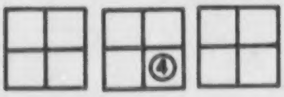
$= 4 \times 2 = 8$



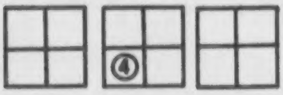
$= 4 \times 4 = 16$



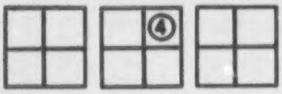
$= 4 \times 8 = 32$



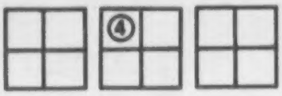
$= 4 \times 10 = 40$



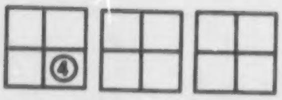
$= 4 \times 20 = 80$



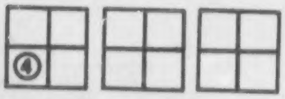
$= 4 \times 40 = 160$



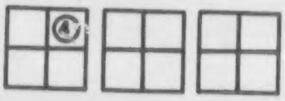
$= 4 \times 80 = 320$



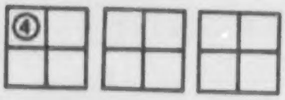
= 4 × 100 = 400



= 4 × 200 = 800



= 4 × 400 = 1,600

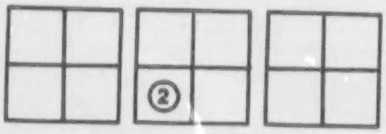


= 4 × 800 = 3,200

Repeat the exercise using the ⑥ -checker.

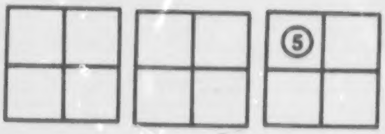
T: I would like someone to put 40 on the Minicomputer using exactly one of the checkers ② through ⑨ .

Accept any solution, for example,



= 40

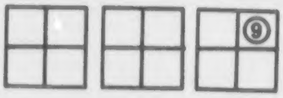
T: Can you put 40 on the ones' board with exactly one of the checkers - ② through ⑨ ?



= 40

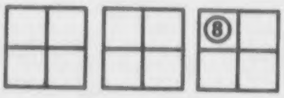
Continue, using only the checkers ② through ⑨ , with

36 with one checker?



= 36

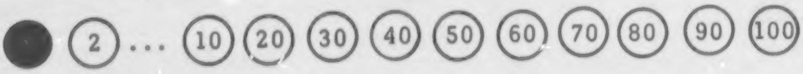
64 with one checker?



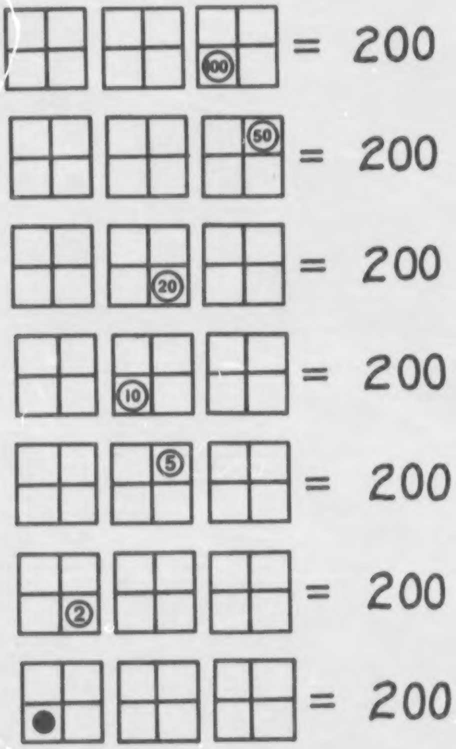
= 64

T: Can you put 200 on the Minicomputer using exactly one checker.

You may use any of these checkers:

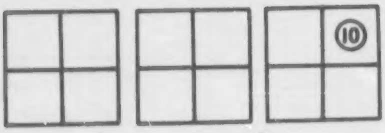

2
...
10
20
30
40
50
60
70
80
90
100

Let students give many solutions to this problem. For example,


100
= 200
50
= 200
20
= 200
10
= 200
5
= 200
2
= 200
= 200

Exercise 2

Display this Minicomputer configuration.

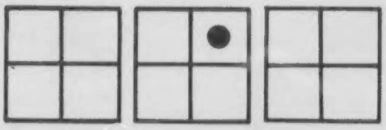


23N

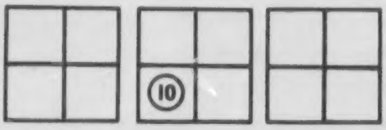
T: What number is this? [Answer: 40]

Can you show me the same number using only a regular checker?

Let a student come to the Minicomputer and make the trade.



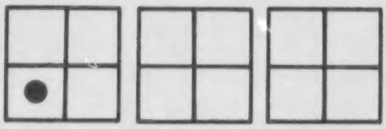
Then display this configuration.



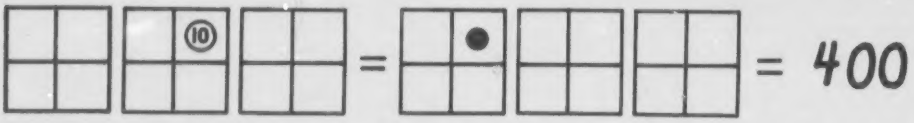
T: What number is this? [Answer: 200]

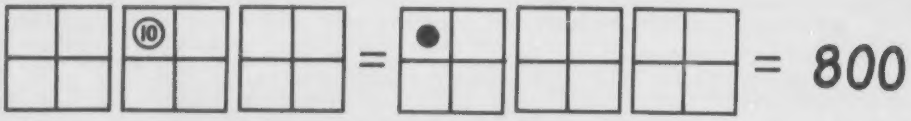
Can you show me the same number using only a regular checker?

Let a student make the trade.



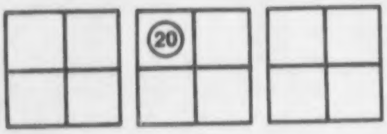
Follow this activity with two more examples.





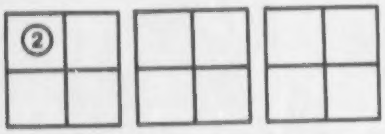
85

Next display this configuration.

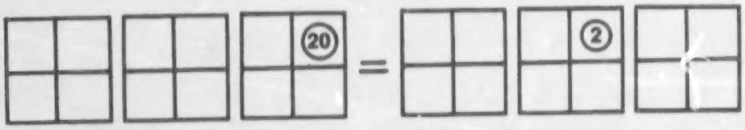


T: What number is this? [Answer: 1,600]

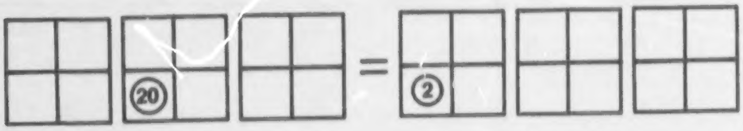
Who can show the same number using exactly one ② -checker?



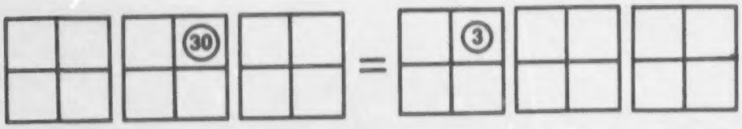
Continue in this way with the following trades.



= 80



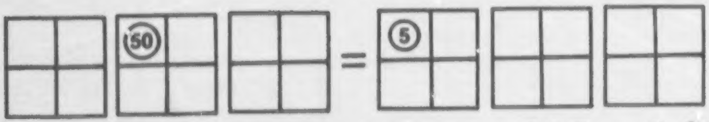
= 400



= 1,200

At this point, reinforce these trades on the Minicomputer by writing a calculation on the board.

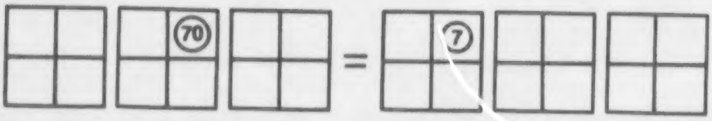
$$30 \times 40 = 3 \times 400 = 1,200$$



= 4,000

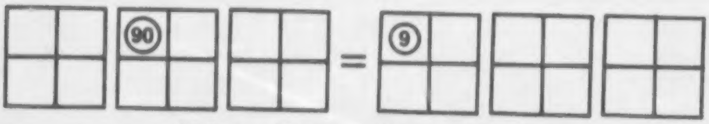
$$50 \times 80 = 5 \times 800 = 4,000$$

23N



= 2,800

70 × 40 = 7 × 400 = 2,800



= 7,200

90 × 80 = 9 × 800 = 7,200

Write these problems on the board and have the students solve them.

20 × 40 = 2 × ☐ = ______

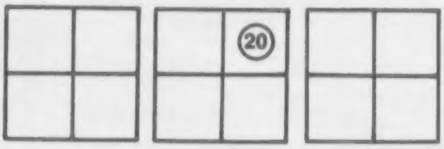
20 × 800 = 2 × ☐ = ______

20 × 64 = 2 × ☐ = ______

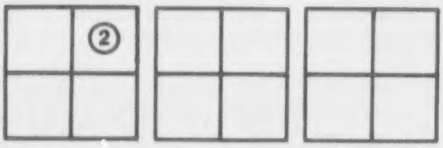
30 × 42 = 3 × ☐ = ______

If there are a few students who have difficulty with these problems, use the Minicomputer to show a trade.

For example,



= 20 × 40



= 2 × 400 = 800

Check the students' work collectively.

$$20 \times 40 = 2 \times \boxed{400} = \underline{800}$$

$$20 \times 800 = 2 \times \boxed{8{,}000} = \underline{16{,}000}$$

$$20 \times 64 = 2 \times \boxed{640} = \underline{1{,}280}$$

$$30 \times 42 = 3 \times \boxed{420} = \underline{1{,}260}$$

Exercise 3

Write this problem on the board.

$$\begin{array}{r} 69 \\ \underline{\times\ 23} \end{array}$$

T: Solve this problem on your paper.

Give the class a few minutes to work the problem, and then work it collectively.

T: First, we multiply by 3. What is 3 × 69?

S: 207.

Write it on the board.

$$3 \times 69 = \quad \begin{array}{r} 69 \\ \underline{\times\ 23} \\ 207 \end{array}$$

T: Next we multiply 69 by 20. What is 20 × 69 or 2 × 690?

S: 1,380.

23N

Write this on the board and complete the problem.

$$\begin{array}{lr} & 69 \\ & \times 23 \\ \hline 3 \times 69 = & 207 \\ 20 \times 69 = 2 \times 690 = & 1{,}380 \\ \hline & 1{,}587 \end{array}$$

Write these problems on the board and ask the students to solve them.

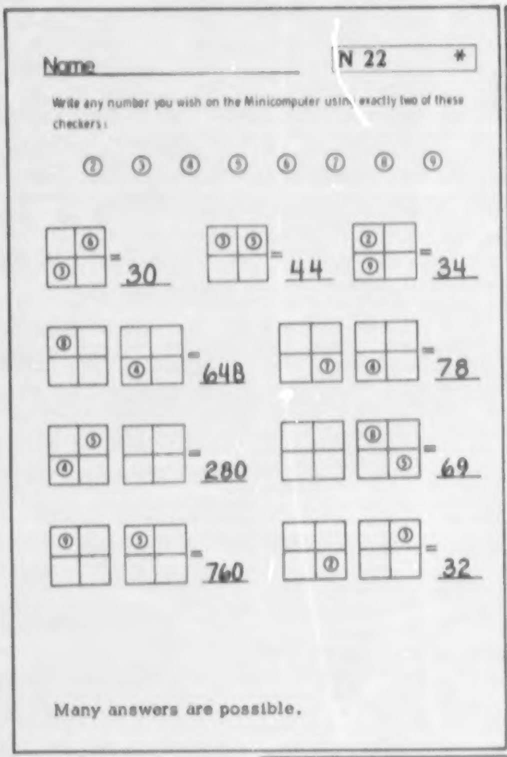
$$\begin{array}{r} 58 \\ \times 23 \\ \hline \end{array} \qquad \begin{array}{r} 74 \\ \times 32 \\ \hline \end{array} \qquad \begin{array}{r} 47 \\ \times 46 \\ \hline \end{array}$$

Have three students come to the board and work the problems while the rest of the class checks their work.

$$\begin{array}{r} 58 \\ \times 23 \\ \hline 1{,}334 \end{array} \qquad \begin{array}{r} 74 \\ \times 32 \\ \hline 2{,}368 \end{array} \qquad \begin{array}{r} 47 \\ \times 46 \\ \hline 2{,}162 \end{array}$$

Worksheets N22, N23, and N24 are available.

Note to the Teacher: Exercise 3 may be modified and repeated as a warm-up exercise to be used whenever time permits. This same activity could be used for a few minutes during free time.



Name ____________ **N 22** *

Write any number you wish on the Minicomputer using exactly two of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

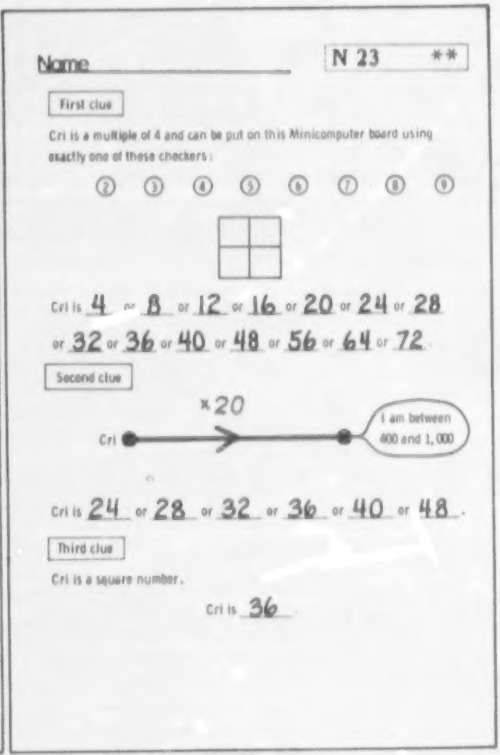
= 30 = 44 = 34

= 648 = 78

= 280 = 69

= 760 = 32

Many answers are possible.



Name ____________ **N 23** **

First clue

Cri is a multiple of 4 and can be put on this Minicomputer board using exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

Cri is 4 or 8 or 12 or 16 or 20 or 24 or 28 or 32 or 36 or 40 or 48 or 56 or 64 or 72.

Second clue

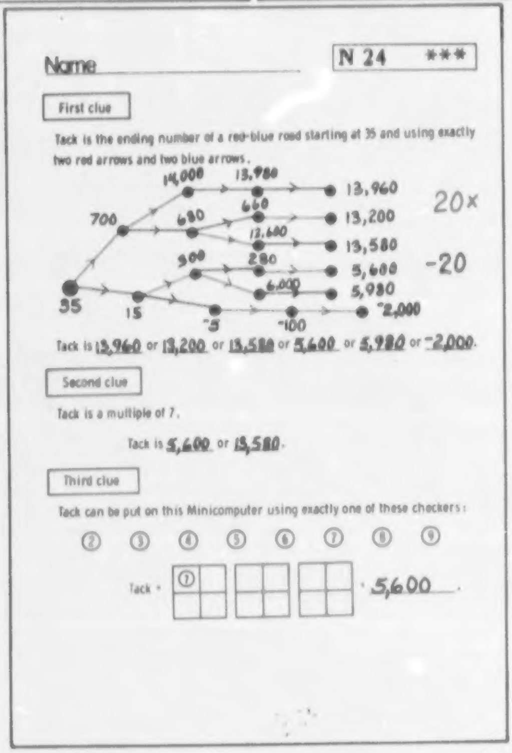
×20
Cri
I am between 400 and 1,000

Cri is 24 or 28 or 32 or 36 or 40 or 48.

Third clue

Cri is a square number.

Cri is 36



Name ____________ **N 24** ***

First clue

Tack is the ending number of a red-blue road starting at 35 and using exactly two red arrows and two blue arrows.

14,000
13,980
13,960
700
680
660
13,200
12,600
13,580
300
280
5,600
6,000
5,980
35
15
⁻5
⁻100
⁻2,000
20×
-20

Tack is 13,960 or 13,200 or 13,580 or 5,600 or 5,980 or ⁻2,000.

Second clue

Tack is a multiple of 7.

Tack is 5,600 or 13,580.

Third clue

Tack can be put on this Minicomputer using exactly one of these checkers:

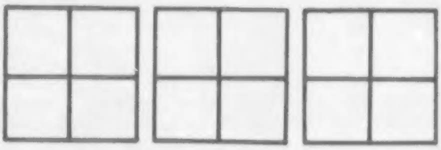
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

Tack = 5,600.

2.7 Multiplication Algorithm #3

Exercise 1

Display three Minicomputer boards

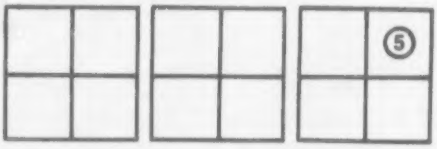


and your set of these checkers:

② • • • ⑩ ⑳ ㉚ ㊵ ㊿ (60) (70) (80) (90) (100)

T: We will use these checkers. I will put a number on the Minicomputer and you tell me what it is.

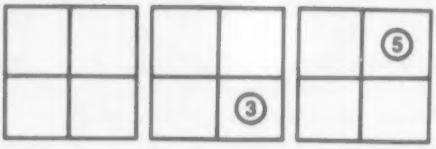
Display a ⑤ -checker.



T: What number is this?

S: 20.

Add this checker.



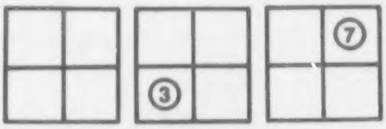
T: What is this number?

S: 50.

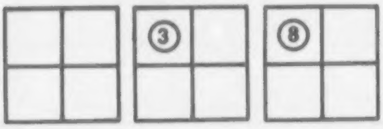
T: How did you get 50?

S: 5 x 4 = 20 and 3 x 10 = 30 and 20 + 30 = 50.

Continue this exercise with these configurations.



= 88

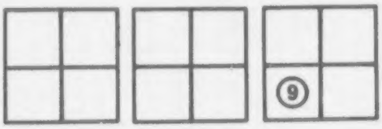


= 304

Clear the Minicomputer of checkers.

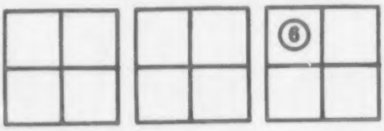
T: Who can put on 18 with exactly one checker?

Let a student put a solution on the Minicomputer.



= 18

T: 48 with exactly one checker?

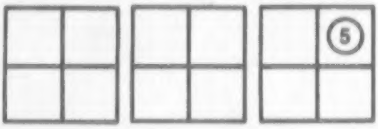


= 48

Exercise 2

T: Who can show 40 using only a ⑤-checker and a ②-checker?

Let a student place one of the checkers. Suppose the ⑤-checker is placed first.



T: What number is on the Minicomputer?

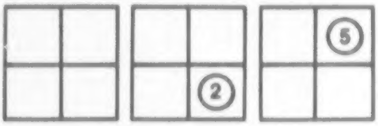
S: 20.

T: Our goal is 40, how much more do we need?

S: 20.

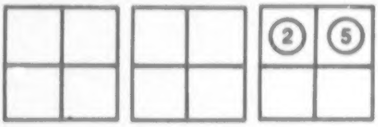
T: Who can place the second checker?

Have another student place the second checker.

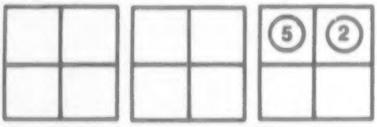


= 40

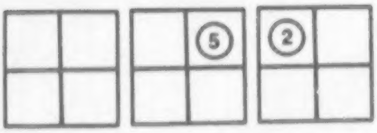
Continue in this way with 36, 48, and 216.



= 36



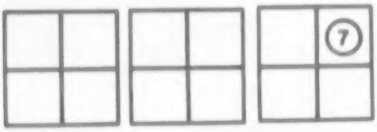
= 48



= 216

T: Who can put on 34 using only a ③ -checker and a ⑦ -checker?

Let a student place one of the checkers, for example the ⑦ -checker.



T: What number is on the Minicomputer?

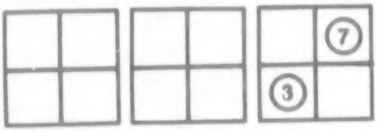
S: 28.

T: How much more do we need?

S: 6.

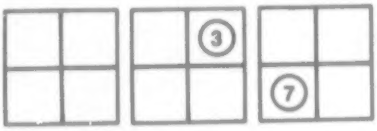
T: Who can place the second checker?

Have another student place the second checker.

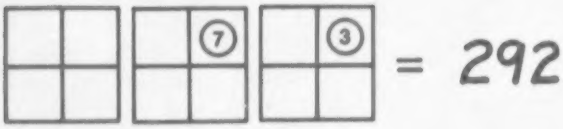


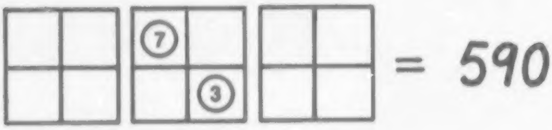
= 34

Continue this exercise with 134; 292; and 590.



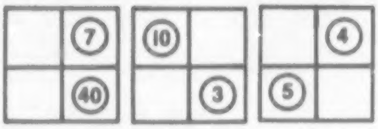
= 134





Exercise 3

Display this configuration.



T: I have put a number on the Minicomputer. Do not calculate the number; just estimate it. Is this number more or less than 1,000?

S: It is more than 1,000, because 7 × 400 = 2,800 (or 40 × 100 = 4,000).

T: Is it more or less than 5,000?

S: More, because 40 × 100 = 4,000 and 4,000 + 2,800 = 6,800.

T: I will do something to this number by moving or trading some checkers. Each time you are to tell me if I get a larger number, a smaller number, or the same number.

Move the ④-checker from the 4-square to the 8-square.

T: What do you think?

S: The number is larger.

T: Write how much larger on your paper.

It is likely that some students will answer that the number is four larger. (In fact, it is 16 larger.)

In this event, ask:

T: If I moved one regular checker from the 4-square to the 8-square, how much larger would the number be?

S: 4 larger.

T: But, I moved a ④ -checker (or four regular checkers) from the 4-square to the 8-square. How much bigger is the number?

S: 16.

T: Why?

S: 4 x 4 = 16.

Continue in this manner making the following moves:

Move a ⑩ -checker from the 80-square to the 40-square. (400 smaller, because 400 = 10 x 40.)

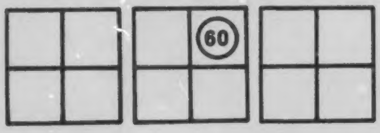
Remove the ⑩ -checker from the 40-square and place a regular checker on the 400-square. (Same)

Remove the ③ -checker on the 10-square and place a ㉚ -checker on the 1-square. (Same)

Move a ⑦ -checker from the 400-square to the 200-square. (1,400 smaller, because 1,400 = 7 × 200.)

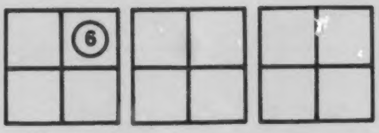
Exercise 4

Display the Minicomputer with this (60) -checker.



T: Who can trade this (60) -checker for a ⑥ -checker and not change the number?

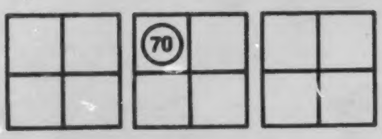
Let a student make the trade.



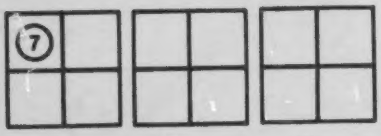
Emphasize the trade by writing this calculation on the board.

$$60 \times 40 = 6 \times 400 = 2,400$$

Clear the Minicomputer and place a (70) -checker on the 80-square.



T: Who can trade this (70) -checker for a ⑦ -checker and not change the number?



Write this calculation on the board.

$$70 \times 80 = 7 \times 800 = 5{,}600$$

Then write these problems on the board and let the class solve them.

$$30 \times 25 = 3 \times \boxed{} = \underline{}$$
$$40 \times 62 = 4 \times \boxed{} = \underline{}$$

Check them collectively.

$$30 \times 25 = 3 \times \boxed{250} = \underline{750}$$
$$40 \times 62 = 4 \times \boxed{620} = \underline{2{,}480}$$

Exercise 5

Write this problem on the board.

$$\begin{array}{r} 47 \\ \times\, 34 \\ \hline \end{array}$$

T: Do this calculation on your paper.

Let your students have a few minutes to work on the problem, then solve it collectively.

T: First, we multiply by 4. What is 4 × 47?

S: 188.

Write it on the board.

$$\begin{array}{rr} & 47 \\ & \times 34 \\ \cline{2-2} 4 \times 47 = & 188 \end{array}$$

T: Next we multiply 47 by 30. What is 30 × 47 or 3 × 470?

S: 1,410.

Write this on the board and complete the calculation.

$$\begin{array}{rr} & 47 \\ & \times 34 \\ \cline{2-2} 4 \times 47 = & 188 \\ 30 \times 47 = 3 \times 470 = & 1{,}410 \\ \cline{2-2} & 1{,}598 \end{array}$$

Write these problems on the board and ask the students to do the calculations on their papers.

$$\begin{array}{r} 38 \\ \times 15 \\ \hline \end{array} \qquad \begin{array}{r} 42 \\ \times 29 \\ \hline \end{array} \qquad \begin{array}{r} 86 \\ \times 33 \\ \hline \end{array}$$

Ask three students to do the calculations at the board while the rest of the class checks their work.

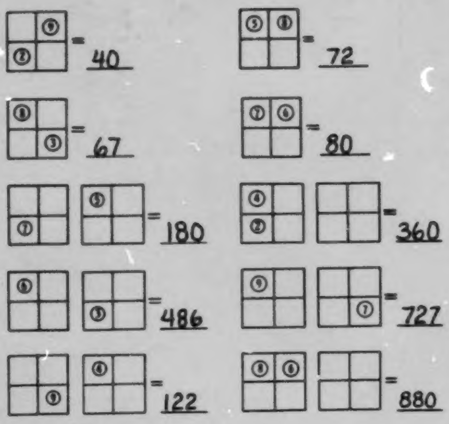
$$\begin{array}{r} 38 \\ \times\ 15 \\ \hline 190 \\ 380 \\ \hline 570 \end{array} \qquad \begin{array}{r} 42 \\ \times\ 29 \\ \hline 378 \\ 840 \\ \hline 1{,}218 \end{array} \qquad \begin{array}{r} 86 \\ \times\ 33 \\ \hline 258 \\ 2{,}580 \\ \hline 2{,}838 \end{array}$$

Worksheets N25, N26, and N27 are available.

Name ____________ **N 25 ***

Write any number you wish on the Minicomputer using exactly two of these checkers:

③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

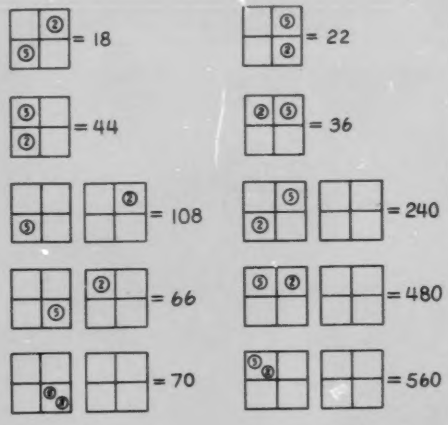

= 40
= 72
= 67
= 80
= 180
= 360
= 486
= 727
= 122
= 880

Many solutions are possible.

Name ____________ **N 26 ****

Write each number on the Minicomputer using exactly these two checkers:

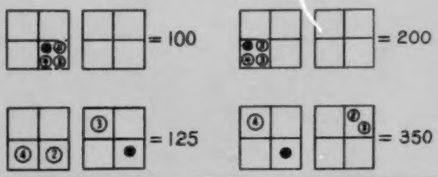
② ⑤


= 18
= 22
= 44
= 36
= 108
= 240
= 66
= 480
= 70
= 560

Name ____________ **N 27 *****

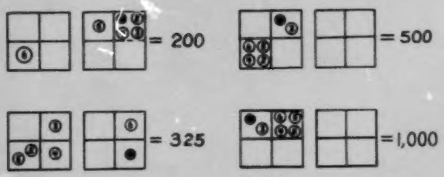
Write each number on the Minicomputer using exactly one positive regular checker and all of these three new checkers.

● ② ③ ④


= 100
= 200
= 125
= 350

Write each number on the Minicomputer using exactly one positive regular checker and all of these five new checkers.

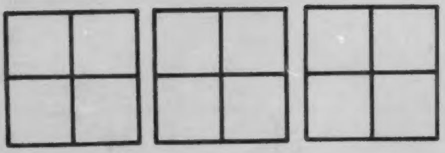
● ② ③ ④ ⑤ ⑥


= 200
= 500
= 325
=1,000

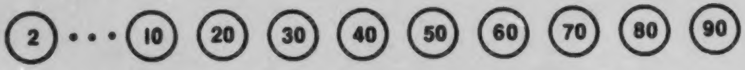
2.8 How Many Ways?

Exercise 1

Display three Minicomputer boards

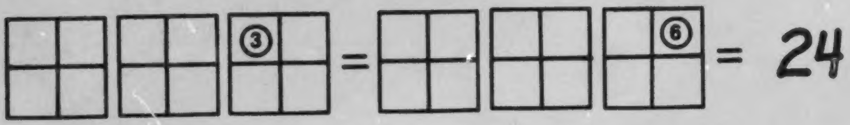


and your set of these checkers:


2
...
10
20
30
40
50
60
70
80
90

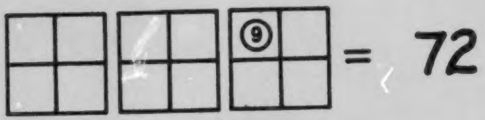
T: Who can put 24 on the Minicomputer using exactly one of these checkers?

Encourage the students to give you both solutions.

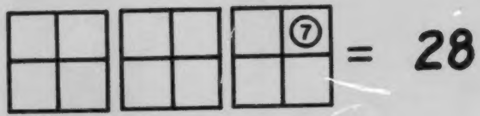


Next ask for

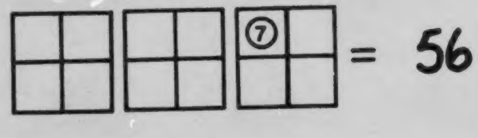
72 with exactly one checker?



28 with exactly one checker?



56 with exactly one checker?

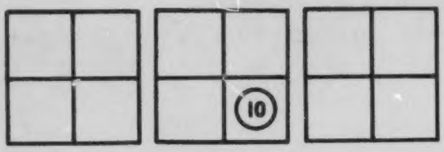


31N

Clear the Minicomputer of checkers.

T: Now, I would like to put on 116 using one ⑩ -checker and exactly one other checker.

Let a student place one checker. Suppose the ⑩ -checker is placed first.



T: What number is this?

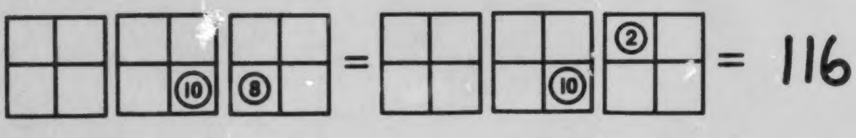
S: 100.

T: The goal is 116; how much more do we need?

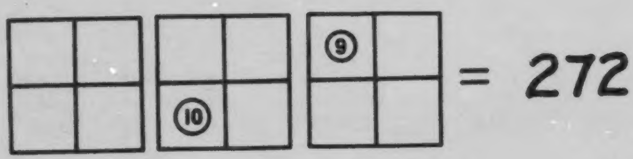
S: 16.

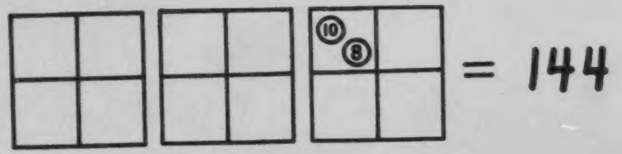
T: Who can place the second checker?

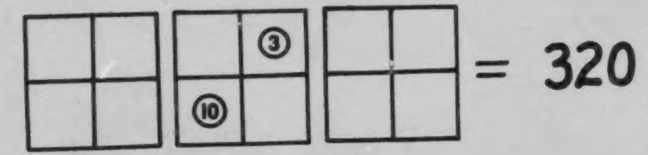
Let a student place the second checker.

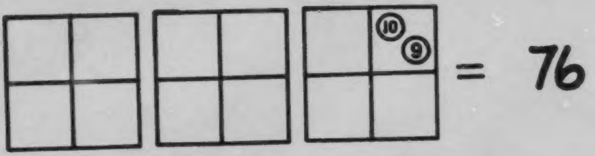


Continue this activity using 272, 144, 320, and 76.



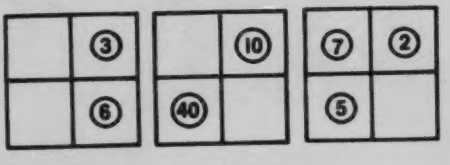






Exercise 2

Put these checkers on the Minicomputer.



T: I have put a number on the Minicomputer. Do not calculate the number; just estimate it. Is this number greater than or less than 1,000?

S: Greater than 1,000. 1,800 is on the hundreds' board.

T: Is this number greater than or less than 3,000?

S: It is greater than 3,000. 1,800 is on the hundreds' board and 1,200 is on the tens' board.

T: Very good. Now, I will do something to this number by moving or trading some checkers. You are to tell me if the new number is larger, smaller, or the same.

Move the ② -checker from the 4 - square to the 8 - square.

T: What do you think?

S: The number is larger.

T: How much larger? Write how much larger on your paper.

It is possible that some student will answer that the number is 4 larger, when, in fact, it is 8 larger. In this event, ask:

T: If I moved one regular checker from the 4 - square to the 8 - square, the number would be 4 larger. But, I moved a ② -checker (which is the same as two regular checkers) from the 4 - square to the 8 - square. How much larger is the number?

S: 8 larger.

T: Why?

S: 2 × 4 = 8.

Continue this activity, making the following moves:

Move the ⑩ -checker from the 40 - square to the 10 - square.

(300 smaller; 300 = 10 × 30)

Move the ⑤ -checker from the 2 - square to the 8 - square.

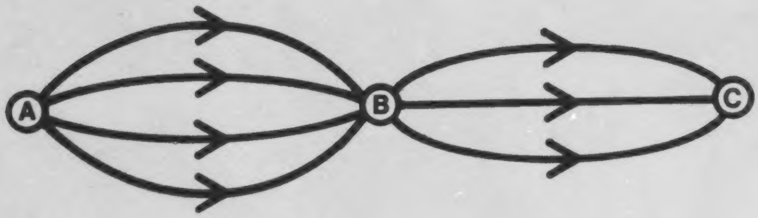
(30 larger; 30 = 5 × 6)

Remove the ③ -checker from the 400 - square and place a ㉚ -checker on the 40 - square. (Same)

Remove the ㊵ -checker from the 20 - square and place a ④ -checker on the 200 - square. (Same)

Exercise 3

Draw this picture on the board.



T: Three cities, A, B, and C are connected by one-way roads. Show me a way to go from city A to city C through city B.

Let several students trace paths from A to C.

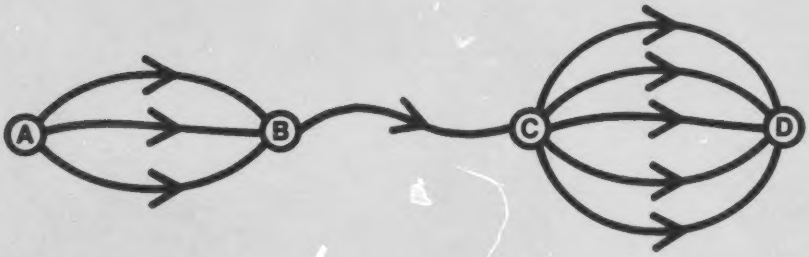
T: How many ways are there to go from A to C?

S: 12.

T: Why?

S: If I follow the top arrow from A to B, I have three ways to go from B to C. The same is true for each of the three other arrows from A to B, so I multiply: 4 × 3 = 12.

Draw this picture on the board.



T: Show me a way to go from A to D.

Again let several students trace paths.

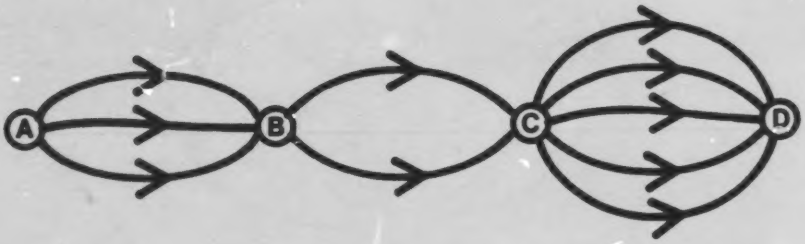
T: How many different ways are there to go from A to D?

S: 15.

T: Explain how you found 15.

S: From A to C there are 3 paths, and from C to D there are 5 paths. Then 3 x 5 = 15.

Add one arrow between B and C.

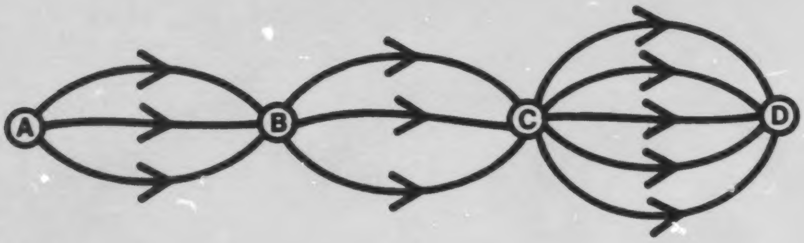


T: Now, how many ways to go from A to D?

S: 30. Add one more arrow and you have 15 more ways.

S: From A to C, 6 ways; so from A to D, 6 x 5 = 30 ways.

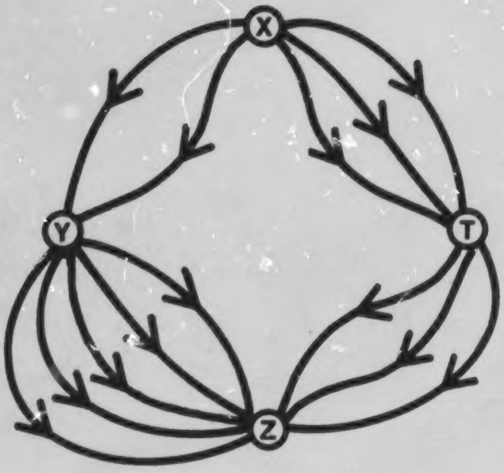
Add one more arrow between B and C.



T: I added one more arrow. How many ways?

S: 15 more; 45.

Now draw this picture.



T: Be careful here. We still have one-way roads. In this situation, we travel from X to Z, but we can travel through Y or through T. Show me a way to go from X to Z.

Let several students trace paths. Be sure that paths through Y and through T are shown.

T: How many ways from X to Z?

If no solution is forthcoming, ask:

T: How many ways to go from X to Z through Y?

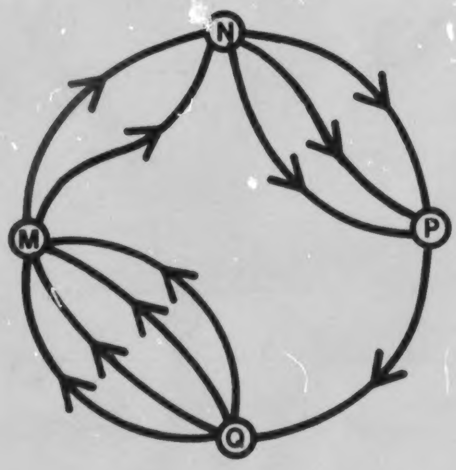
S: 10 ways (2 × 5 = 10).

T: How many ways to go from X to Z through T?

S: 9 (3 × 3 = 9).

T: Then there are 10 + 9 or 19 ways to go from X to Z.

Draw this picture on the board.



T: In this situation, we start at M and arrive back at M, having passed through N, P, and Q. Show me a path I could take.

Let several students trace paths.

T: How many different ways to go from M to M passing through each city?

S: 24, because there are 6 ways (3 × 2 × 1 = 6) from M to Q, then from M to M there are 6 × 4 or 24 ways.

Worksheets N28, N29, and N30 are available.

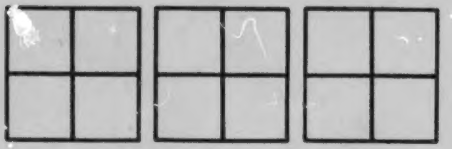
If some of your students finish the worksheets before the end of class, let them solve this 4-star problem which you write on the board.

31N

****

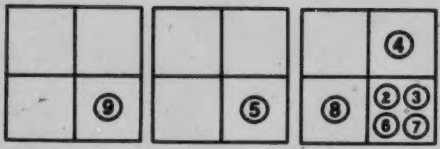
Put 1,000 on the Minicomputer using all of these checkers exactly once.

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

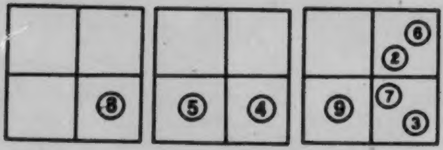


= 1,000

Two possible solutions:



= 1,000

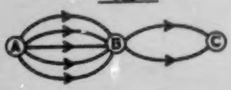


= 1,000

Name ______________________

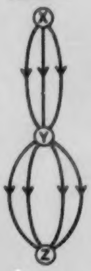
N 28 *

How many roads can you choose to go . . .



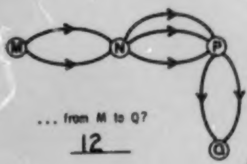
. . . from A to C?

10



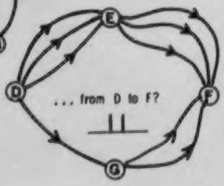
. . . from X to Z?

12



. . . from M to Q?

12



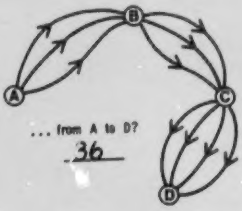
. . . from D to F?

11

Name ______________________

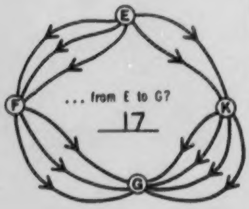
N 29 **

How many roads can you choose to go . . .



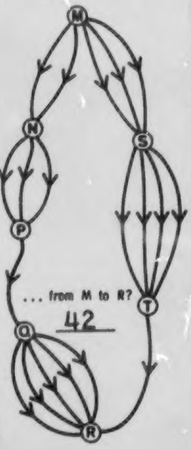
. . . from A to D?

36



. . . from E to G?

17



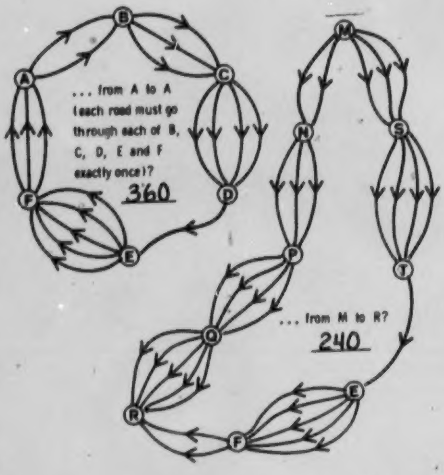
. . . from M to R?

42

Name ______________________

N 30 ***

How many roads can you choose to go . . .



. . . from A to A (each road must go through each of B, C, D, E and F exactly once)?

360

. . . from M to R?

240

3

DECIMAL NUMBERS

This chapter reviews and extends the students' experiences with decimal numbers which began in the CSMP Mathematics for the Upper Primary Grades program. Activities involve finding decimal names for certain fractions, calculating sums of decimal numbers, determining multiples of decimal numbers, and locating points on the number line for decimal numbers in the ten-thousandths.

A new game, the Newtonian Quiz, is introduced. This game focuses attention on the successive decimal graduations of the number line and the order of decimal numbers.

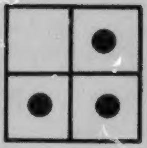
CAPSULE LESSON SUMMARIES

Section 3.1:	Review decimal numbers, using the Minicomputer, the number line, and the hand-calculator. (Worksheets N31, N32, N33)
Section 3.2:	Solve a detective story for which the secret number could be a decimal number, where the clues involve the hand-calculator, string pictures, arrow pictures, and the Minicomputer. (worksheets N34, N35)
Section 3.3:	Find a decimal name for a given fraction with the help of the number line. Calculate the sum of decimal numbers using arrow pictures and the hand-calculator as aids. (Worksheets N36, N37, N38)
Section 3.4:	Extend the experiences of Section 3.3 to include decimal names for numbers such as $1\frac{1}{5}$, and calculations such as "0.6 + 1.3". Play the Newtonian Quiz game, using the number line to focus attention on decimal numbers to the nearest tenth. (Worksheets N39, N40, N41)
Section 3.5:	Play the Newtonian Quiz game with the secret number a decimal number in the thousandths. Then play a version of Minicomputer Golf for which the starting number on the Minicomputer is a decimal number. (Worksheets N42, N43, N44)
Section 3.6:	Find some multiples of a given number and then some multiples of the multiples (using an arrow picture situation to state the problems). Find some numbers, including decimal numbers, having the given number as a multiple. (Worksheets N45, N46, N47)
Section 3.7:	Play the Newtonian Quiz game with the secret number in the ten-thousandths. Find midpoints of pairs of numbers on the number line. Given a number, find two numbers having the given number as midpoint. (Worksheets N48, N49, N50, N51)

3.1 Decimal Numbers and the Minicomputer

Exercise 1

Display one Minicomputer board.

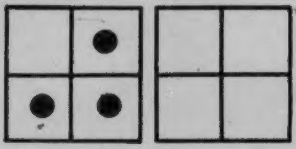


T: Who can tell me an addition problem they see on the Minicomputer?

S: 4 + 3 = 7.

S: 3 + 4 = 7.

Write these addition problems on the chalkboard. Then add one board to the right.

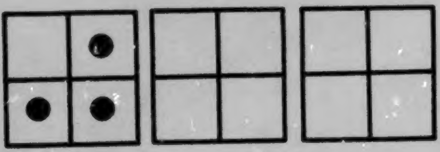


T: Who can tell me an addition problem they see now?

S: 40 + 30 = 70.

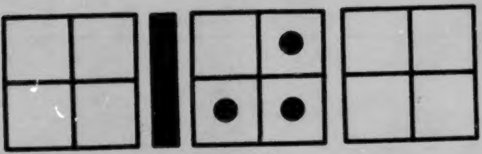
S: 30 + 40 = 70.

Write these on the chalkboard. Continue in this manner first adding one more board to the right,



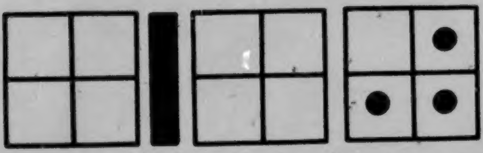
400 + 300 = 700
or
300 + 400 = 700

and then moving the boards to this position.



0.4 + 0.3 = 0.7
or
0.3 + 0.4 = 0.7

Next reverse the dimes' and pennies' boards.



0.04 + 0.03 = 0.07
or
0.03 + 0.04 = 0.07

Repeat this exercise for

8 + 4 = 12

80 + 40 = 120

800 + 400 = 1,200

0.8 + 0.4 = 1.2

0.08 + 0.04 = 0.12

It is likely that the problem "0.8 + 0.4 = 1.2" will give the class some trouble. In our development classes, a frequent response was "0.12". If the problem is rephrased in terms of money (i.e., 8 dimes plus 4 dimes or 80 cents plus

40 cents), students very quickly see that 0.8 + 0.4 = 1.2. Write the following problems on the board and let the students solve them. Encourage them to make use of the patterns they have just practiced with the Minicomputer.

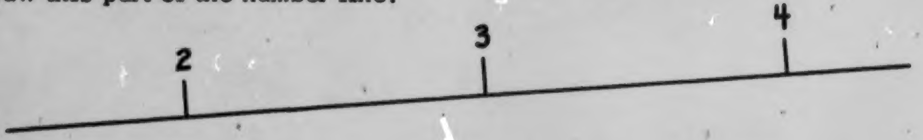
$$37 + 25 = \boxed{\quad}$$
$$370 + 250 = \boxed{\quad}$$
$$3.7 + 2.5 = \boxed{\quad}$$
$$0.37 + 0.25 = \boxed{\quad}$$

After a while, check the answers as a class.

$$37 + 25 = \boxed{62}$$
$$370 + 250 = \boxed{620}$$
$$3.7 + 2.5 = \boxed{6.2}$$
$$0.37 + 0.25 = \boxed{0.62}$$

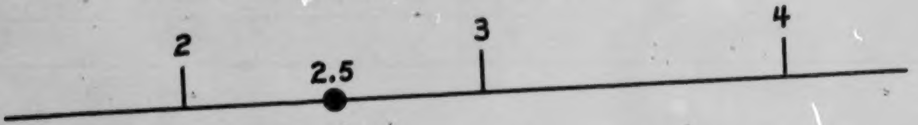
Exercise 2

Draw this part of the number line.



T: Where is 2.5 on the number line?

Let a student put a dot in the approximate location.

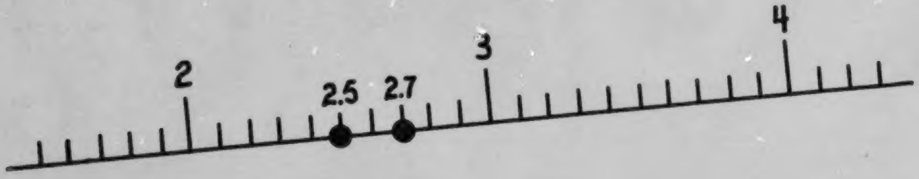


T: Yes, 2.5 is halfway between 2 and 3. It is the midpoint of the segment whose endpoints are 2 and 3. Where is 2.7?

Let a student point to the approximate location.

T: What could we do to locate 2.7 better?

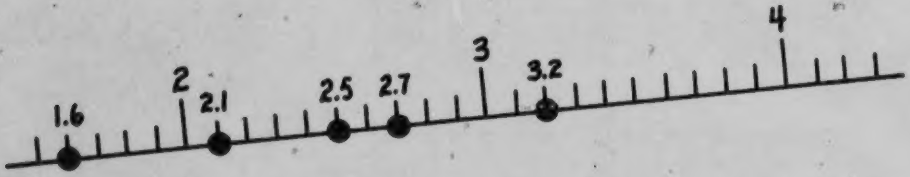
Very likely some students will suggest that you divide the segment between 2 and 3 into ten smaller segments of the same length. Do so and continue subdividing until you have this part of the number line.



T: Where is 2.1? 3.2?

Where is 1.6?

If 1.6 gives some trouble, encourage your students to count backward from 2. Your picture should look something like this:



Exercise 3*

Give yourself and each student a hand-calculator.

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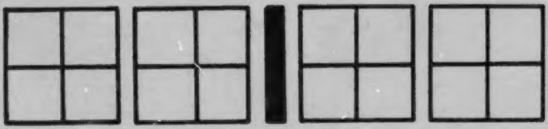
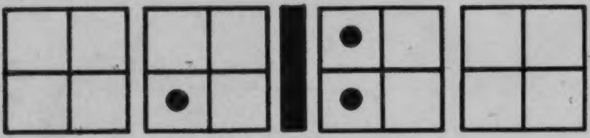
* Exercise 3 may be done without using calculators, if you prefer. Change the instructions by having the students write the numbers on scratch paper.

Display four Minicomputer boards.

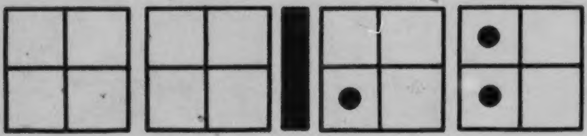


T: I will put a number on the Minicomputer. Try to put the number on the display.

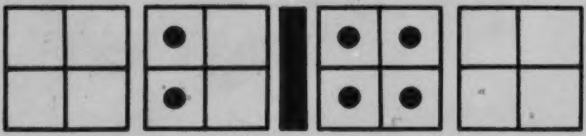
Let the class decode the following configurations. (The answers at the right should not be written on the board until the students give them.)



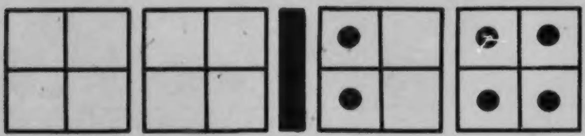
= 3 = 3.0



= 0.3 = 0.30



= 11.50 = 11.5

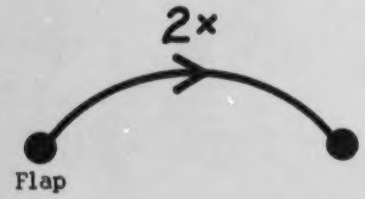


= 1.15

Exercise 4[*]

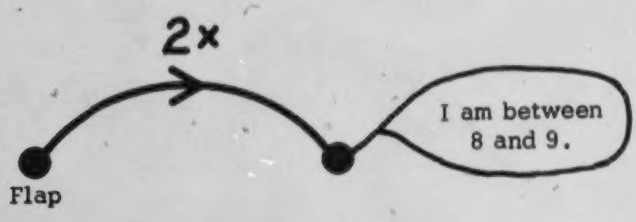
T: Here is a number. Let's call it "Flap". (Draw a dot and label it "Flap".) I will put Flap on the display. Now I press $\boxed{x}$ $\boxed{2}$ $\boxed{=}$.

Draw a 2 x arrow in your picture.



T: The new number I get is between 8 and 9.

Add this information to your arrow picture.



T: Write on your paper some numbers that Flap could be.

Let the students use their calculators to explore numbers that Flap could be. You will be surprised by the variety of numbers that the students will suggest. In our development classes, the following suggestions were made for Flap: 4.1; 4.2; 4.4; 4.31; 4.33; and 4.45.

Worksheets N31, N32, and N33 are available for the rest of the lesson.

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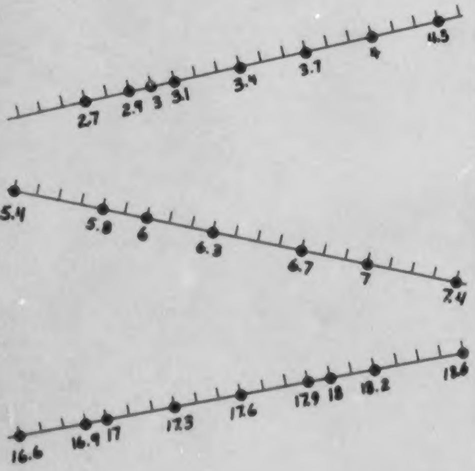
[*] Exercise 4 may be done without the students using calculators, if you prefer.

Name ____________ **N 31** *

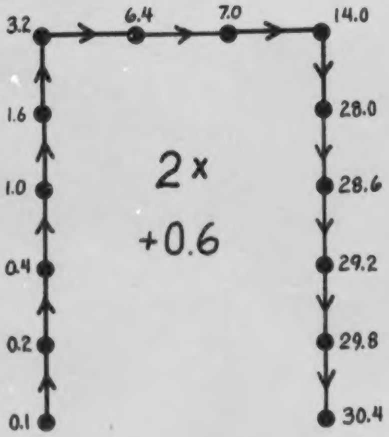
Label the dots.


2.7
2.9
3
3.1
3.4
3.7
4
4.3
5.4
5.8
6
6.3
6.7
7
7.4
16.6
16.9
17
17.3
17.6
17.9
18
18.2
18.6

Name ____________ **N 32** **

First clue

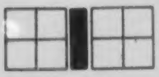
Flip is in this picture.


3.2
6.4
7.0
14.0
1.6
1.0
0.4
0.2
0.1
28.0
28.6
29.2
29.8
30.4
2x
+0.6

Name ____________ **N 33** ***

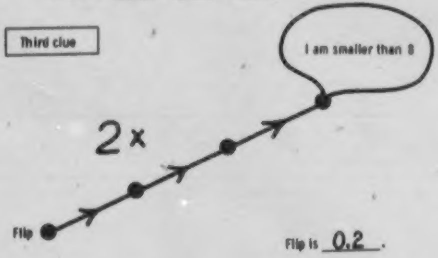
Second clue

Flip can be put on the Minicomputer using exactly one checker (positive or negative) on each board.



Flip could be 0.2 or 1.6 or 3.2.

Third clue


I am smaller than 8
2x
Flip

Flip is 0.2.

3.2 A Detective Story about Zip

Distribute as many hand-calculators as you have available.

First clue

T: Today we have a detective story about a secret number called "Zip". If I put Zip on the display and press $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ and so on, 12 will appear on the display. What numbers could Zip be? Try to think of some decimal numbers,* too.

Give the students time to explore the numbers that Zip could be. When a number for Zip is suggested, have the students enter the number on their displays and check to see if it could be Zip.

S: Zip could be 3.

T: Let's check it. Put 3 on your display and press $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ and so on. Does 12 appear?

S: Yes.

T: How many times did you have to press $\boxed{=}$?

S: Four times.

Continue in this manner for many numbers that Zip could be, making a list on the board of these numbers as they are found. In our development classes, the students found that Zip could be 12; 6; 4; 3; 2; 1.5; 1; 0.8; 0.75; 0.6; 0.3; 0.2; 0.04; 0.02; 0.01; etc.

---

* What we mean here is "non-integer decimal numbers" (e.g., 1.5; 0.15; etc.), but this expression is much too cumbersome at this level, so we say simply "decimal numbers".

While the class is finding numbers that Zip could be, you might want to try this exercise.

S: Zip could be 0.2.

T: Put 0.2 on your display. Press $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ and watch what happens. What number is on the display?

S: 1.0.

T: Suppose we press $\boxed{=}$ five more times. What do you predict will be on the display?

S: 2.0.

T: Try it and see if you are correct. Now hide the display. Press $\boxed{=}$ until you think you have 4 on the display. How many times did you press $\boxed{=}$?

S: Ten times.

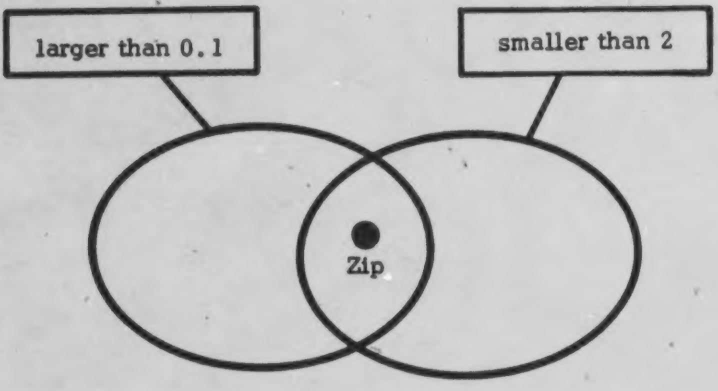
T: Suppose you keep pressing $\boxed{=}$. Will 12 appear?

S: Yes.

This exercise can be repeated with other numbers as long as there is student interest.

Second clue

Draw this string picture.



T: Zip is in this string picture. What does the picture tell us about Zip?

S: Zip is larger than 0.1 and smaller than 2.

T: Give me a number from our list that Zip cannot be. (Accept any number that is less than or equal to 0.1 or greater than or equal to 2; for example, 0; 0.05; 2; 4; 6. Cross that number off the list.)

T: Who can tell me a number Zip can be?

S: Any number from the list that is larger than 0.1 and smaller than 2. (E.g., 1.5; 1; 0.8; 0.75; 0.6; 0.3; 0.2; etc.)

Third clue

Distribute worksheet N34. Walk around the room monitoring student progress. Worksheet N35 is available for those students that finish quickly. After most of the class have finished the N34 worksheet, draw the arrow picture from N34 on the board and have student volunteers label the dots. The completed picture can be found at the end of this section. From the work at the board, one can see that Zip can be 0.25; 0.75; 0.5; 1.0; or 1.5. It is possible that some of these numbers will be overlooked. Remind the class of the first clue.

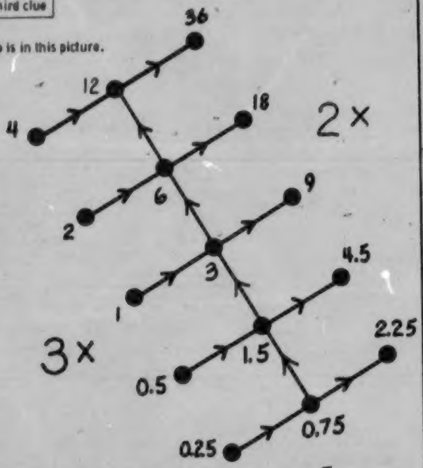
Fourth clue

It is very likely that not everyone will work independently on Worksheet N35. Be sure each student has a copy of Worksheet N35. Then, using the class demonstration Minicomputer, let volunteers come up and make the necessary robot moves to determine each number. The class should conclude that Zip is 1.5.

Name ______________________ **N 34** *

Third clue

Zip is in this picture.

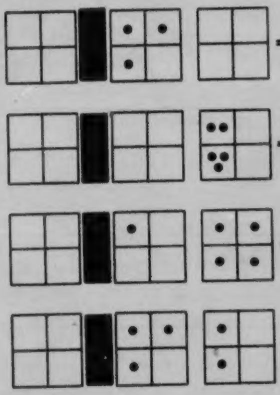

36
12
18
4
2×
6
9
2
3
4.5
1
3×
1.5
2.25
0.5
0.75
0.25

Zip could be 0.25 or 0.75 or 0.5 or 1 or 1.5.

Name ______________________ **N 35** **

Fourth clue

Zip is one of these numbers.



= 1.4

= 0.22

= 0.95

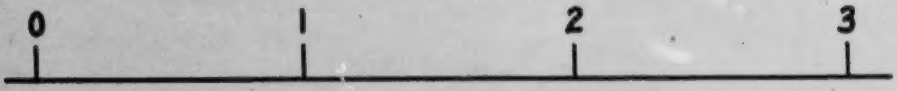
= 1.5

Zip is 1.5.

3.3 Decimals and the Number Line #1

Exercise 1

Draw this part of the number line on the board.



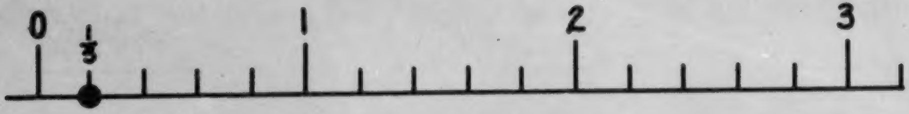
T: Let's put $\frac{1}{5}$ on the number line. Where can we locate a dot for $\frac{1}{5}$?

(Answer: Between 0 and 1)

It is possible that some students will suggest that $\frac{1}{5}$ is between 1 and 2. Should this happen tell them $1\frac{1}{5}$ (Read: one and one-fifth) is between 1 and 2 but that you are trying to locate $\frac{1}{5}$. Let a student put a dot on the number line for the (approximate) location of $\frac{1}{5}$.

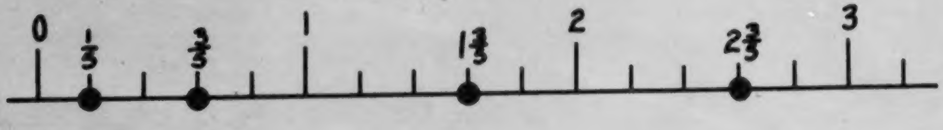
T: What could we do to locate $\frac{1}{5}$ even better? (Answer: Divide the segment between 0 and 1 into five equal smaller segments).

Do this not only for the segment between 0 and 1 but also for the rest of the number line. Then have a student place a dot for $\frac{1}{5}$.

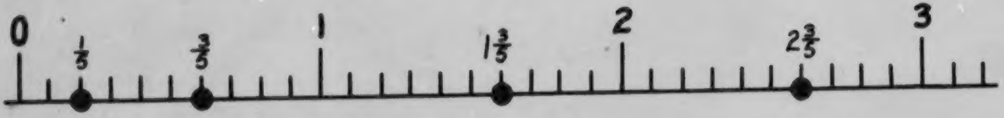


T: Now who can put a dot on the number line for $\frac{3}{5}$? for $1\frac{3}{5}$? for $2\frac{3}{5}$?

Let your students label these points. Your picture should look something like this:



Add tenths graduations to your number line.


0
$\frac{1}{5}$
$\frac{3}{5}$
1
$1\frac{3}{5}$
2
$2\frac{3}{5}$
3

T: Who can give me another name for $\frac{1}{5}$?

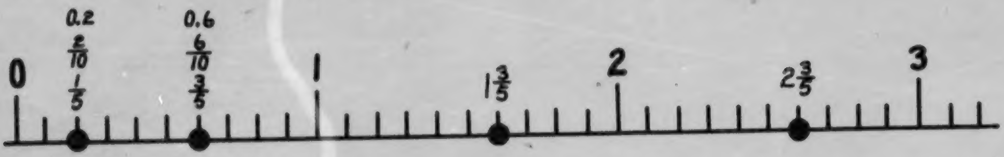
S: "0.2".

S: "$\frac{2}{10}$".

Write these names on the board.

$$\frac{1}{5} = \frac{2}{10} = 0.2$$

Do a similar exercise for $\frac{3}{5}$. ($\frac{3}{5} = \frac{6}{10} = 0.6$)

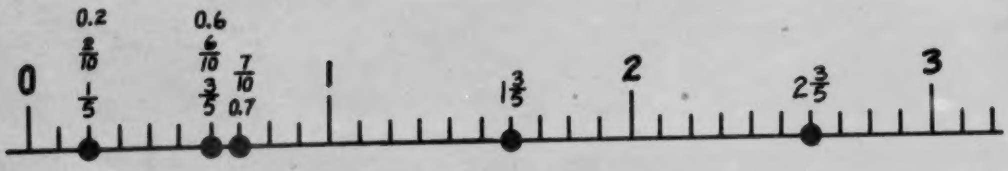

0.2
$\frac{2}{10}$
$\frac{1}{5}$
0.6
$\frac{6}{10}$
$\frac{3}{5}$
0
1
$1\frac{3}{5}$
2
$2\frac{3}{5}$
3

T: Where do we locate 0.7 on the number line? (Answer: Between 0 and 1)

Let someone label the point for 0.7.

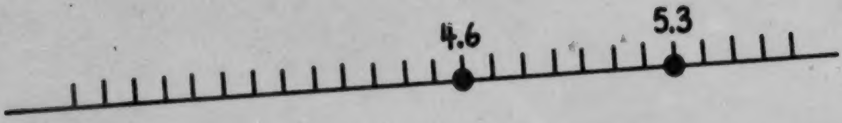
T: Can you think of a fractional name for 0.7?

S: "$\frac{7}{10}$".


0.2
$\frac{2}{10}$
$\frac{1}{5}$
0.6
$\frac{6}{10}$
$\frac{3}{5}$
$\frac{7}{10}$
0.7
0
1
$1\frac{3}{5}$
2
$2\frac{3}{5}$
3

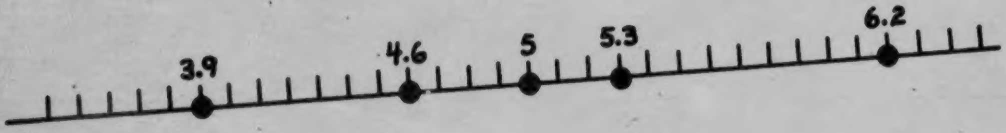
Exercise 2

Draw this part of the number line on the board.

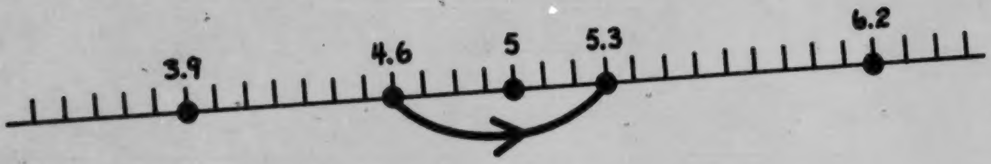


T: I have put 4.6 and 5.3 on the number line. Who can show me 5? 3.9? 6.2?

Let volunteers come up and locate each number in turn. Should anyone have difficulty, it may be necessary to let the class count aloud (forward and backward) on the number line.



Now add this arrow to your picture.



T: What could this red arrow be for?

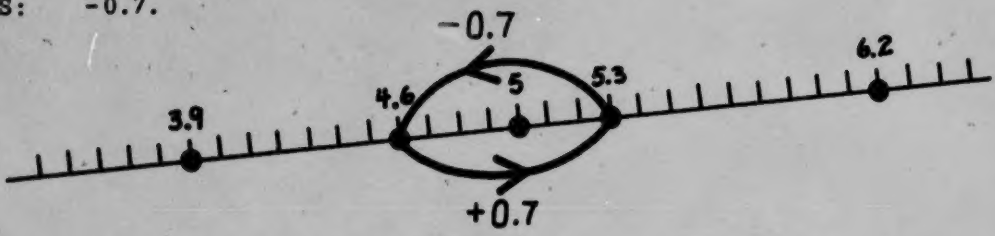
S: +0.7.

T: Let's write the calculation.

$$\begin{array}{r} 4.6 \\ +\,0.7 \\ \hline 5.3 \end{array}$$

T: If I draw the return arrow, what could it be for?

S: -0.7.

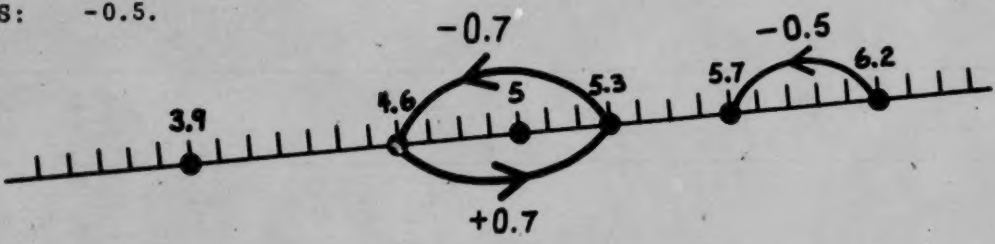


T: Who can show us where to locate 5.7?

Let a student come to the board and point to the location of 5.7.

T: If I draw this arrow (black in the picture below; use any third color), what could it be for?

S: -0.5.



T: Let's write the calculation.

$$\begin{array}{r} \overset{5}{\cancel{6}}.\overset{10}{2} \\ -0.5 \\ \hline 5.7 \end{array} \quad \text{OR} \quad \begin{array}{r} 6.\overset{10}{2} \\ -\overset{1}{0}.5 \\ \hline 5.7 \end{array}$$

Exercise 3

Distribute as many hand-calculators as you have available.

T: Turn on your calculators. Press $\boxed{2}$ $\boxed{\times}$ $\boxed{0}$ $\boxed{.}$ $\boxed{5}$ $\boxed{+}$ $\boxed{9}$ $\boxed{=}$.

What number is on the display?

S: 10.

T: Clear the display. Cover the display and press

$\boxed{2}\ \boxed{\times}\ \boxed{6}\ \boxed{+}\ \boxed{0}\ \boxed{\cdot}\ \boxed{7}\ \boxed{=}$.

What number should be on the display?

S: 12.7.

T: Check the display. Now clear and cover your display.

Press $\boxed{2}\ \boxed{\times}\ \boxed{0}\ \boxed{\cdot}\ \boxed{6}\ \boxed{+}\ \boxed{0}\ \boxed{\cdot}\ \boxed{3}\ \boxed{=}$.

What number should be on the display?

S: 1.5.

Try a similar exercise for

$\boxed{2}\ \boxed{\times}\ \boxed{0}\ \boxed{\cdot}\ \boxed{8}\ \boxed{+}\ \boxed{0}\ \boxed{\cdot}\ \boxed{4}\ \boxed{=}$ (Answer: 2.0)

$\boxed{3}\ \boxed{\times}\ \boxed{0}\ \boxed{\cdot}\ \boxed{6}\ \boxed{+}\ \boxed{1}\ \boxed{\cdot}\ \boxed{2}\ \boxed{=}$ (Answer: 3.0)

T: Clear your display and follow my directions.

Press $\boxed{0}\ \boxed{\cdot}\ \boxed{4}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}$.

What number is on the display?

S: 2.4.

T: Now cover your display. Press $\boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}$.

What number should be on the display?

S: 4.0.

T: Clear the display. Press $\boxed{0}$ $\boxed{\cdot}$ $\boxed{6}$ $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$.

What number is on the display?

S: 1.8.

T: Now hide the display and press $\boxed{=}$ $\boxed{=}$ $\boxed{=}$.

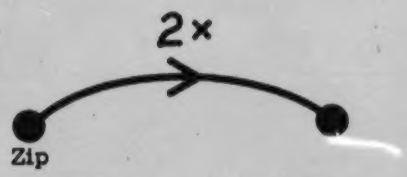
What number should be on the display?

S: 3.6.

Exercise 4

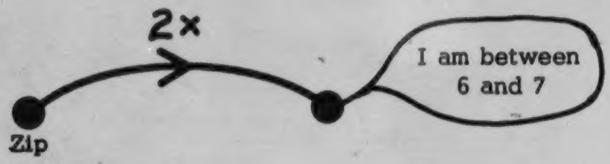
T: I have a secret number on the display of my calculator. I will call it "Zip". (Draw a dot and label it "Zip".) I press $\boxed{\times}$ $\boxed{2}$ $\boxed{=}$.

Draw a 2 x arrow.



T: The new number is between 6 and 7.

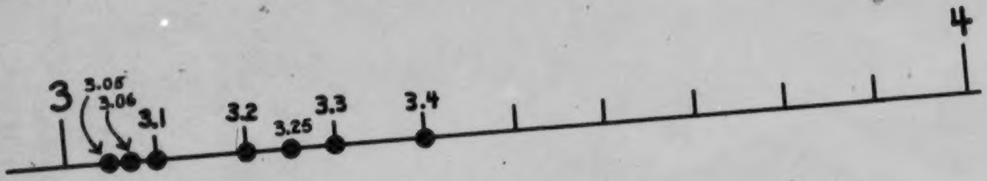
Add this information to your picture.



T: Write on your paper some numbers that Zip could be. You may use your calculators.

Give the students time to explore this situation, and then ask for numbers that Zip could be. In our development class the following suggestions were made for Zip: 3.1; 3.05; 3.3; 3.2; 3.4; 3.06; and 3.25.

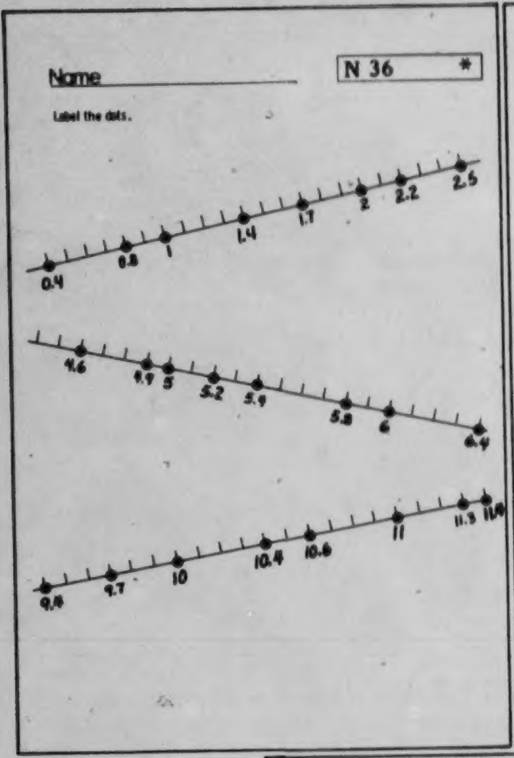
If time permits you might want to locate some of the numbers that Zip could be on the number line. For example,



It is not necessary to subdivide the number line any more than we have done above. For decimals such as 3.05 or 3.06, you need not insist that they be located accurately. At this stage, be content with the right interval. You will deal with this problem more carefully at a later date.

T: Zip is 3.4.

Worksheets N36, N37, and N38 are available for the rest of the lesson. Let each student start with N36; then as the students finish let them work on N37 and N38.

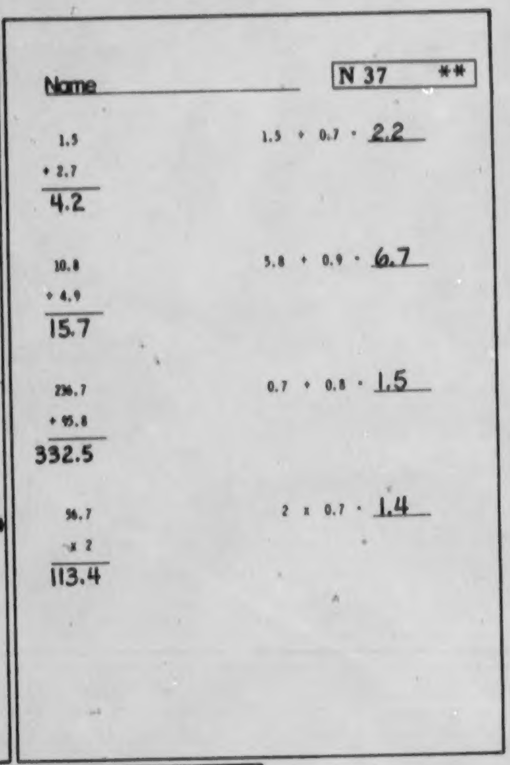


Name ____________

N 36 *

Label the dots.

0.4
0.8
1
1.4
1.7
2
2.2
2.5
4.6
4.9
5
5.2
5.4
5.8
6
6.4
9.4
9.7
10
10.4
10.6
11
11.3
11.4



Name ____________

N 37 **

$$\begin{array}{r} 1.5 \\ +\ 2.7 \\ \hline 4.2 \end{array}$$

1.5 + 0.7 = 2.2

$$\begin{array}{r} 10.8 \\ +\ 4.9 \\ \hline 15.7 \end{array}$$

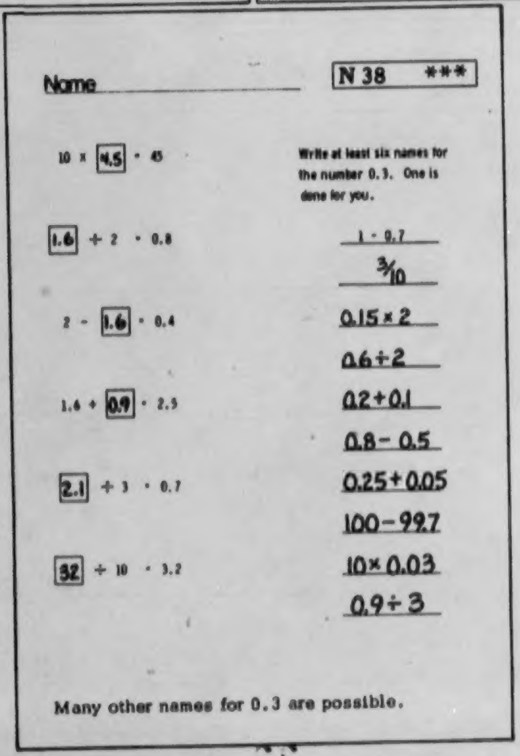
5.8 + 0.9 = 6.7

$$\begin{array}{r} 236.7 \\ +\ 95.8 \\ \hline 332.5 \end{array}$$

0.7 + 0.8 = 1.5

$$\begin{array}{r} 56.7 \\ \times\ 2 \\ \hline 113.4 \end{array}$$

2 x 0.7 = 1.4



Name ____________

N 38 ***

$10 \times \boxed{4.5} = 45$

$\boxed{1.6} \div 2 = 0.8$

$2 - \boxed{1.6} = 0.4$

$1.6 + \boxed{0.9} = 2.5$

$\boxed{2.1} \div 3 = 0.7$

$\boxed{32} \div 10 = 3.2$

Write at least six names for the number 0.3. One is done for you.

1 - 0.7

3/10

0.15 × 2

0.6 ÷ 2

0.2 + 0.1

0.8 − 0.5

0.25 + 0.05

100 − 99.7

10 × 0.03

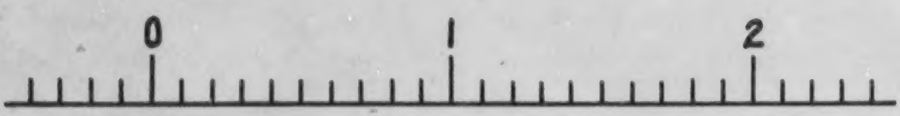
0.9 ÷ 3

Many other names for 0.3 are possible.

3.4 Decimals and the Number Line #2

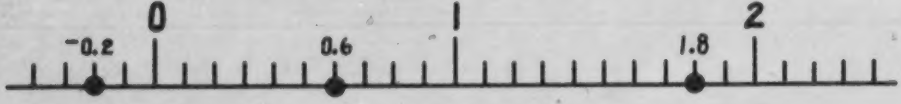
Exercise 1

Draw this part of the number line.



T: Where is 0.6 on the number line?

Let a student come to the board and place a dot for 0.6. Do similar exercises for ⁻0.2 and 1.8. Then your number line should look like this:



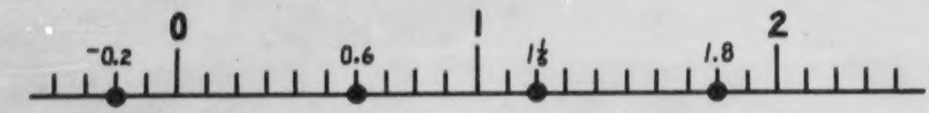
T: Where is $1\frac{1}{5}$ on the number line?

Between which two whole numbers is $1\frac{1}{5}$?

S: Between 1 and 2.

T: Where is $1\frac{1}{5}$ on the number line?

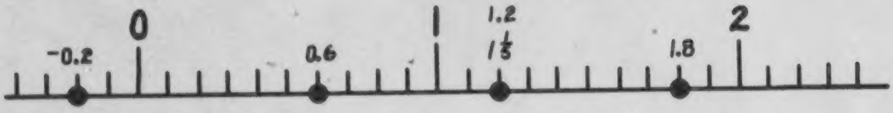
More than likely some students will locate $1\frac{1}{5}$ at the position of 1.5. Remind the students that to locate $1\frac{1}{5}$, the segment between 1 and 2 must be divided into five smaller segments of the same length. Let a student put a dot on the number line for $1\frac{1}{5}$.



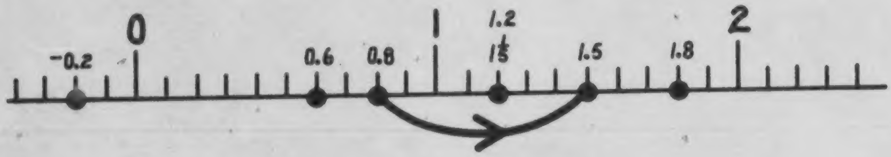
T: What is a decimal name for $1\frac{1}{5}$?

S: "1.2".

Add this information to your number line.


0
1
1.2
$1\frac{1}{5}$
2
⁻0.2
0.6
1.8

Draw this red arrow in your picture.


0
1
1.2
$1\frac{1}{5}$
2
⁻0.2
0.6
0.8
1.5
1.8

T: What could the red arrow be for?

S: +7.

T: Let's check it to see.

Check any answer given by writing the calculation on the board.

$$\begin{array}{r} 0.8 \\ +7.0 \\ \hline 7.8 \end{array}$$

T: Could the red arrow be for +7?

S: No.

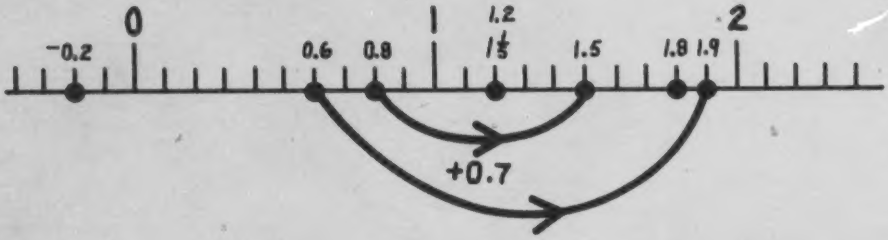
S: +0.7.

T: Let's see. Does 0.8 + 0.7 = 1.5?

S: Yes.

T: Good. Now what could this blue arrow be for?

Draw this blue arrow.


0
1
1.2
2
−0.2
0.6
0.8
1.3
1.5
1.8
1.9
+0.7

S: +0.13.

T: Let's check.

Write the calculation on the board.

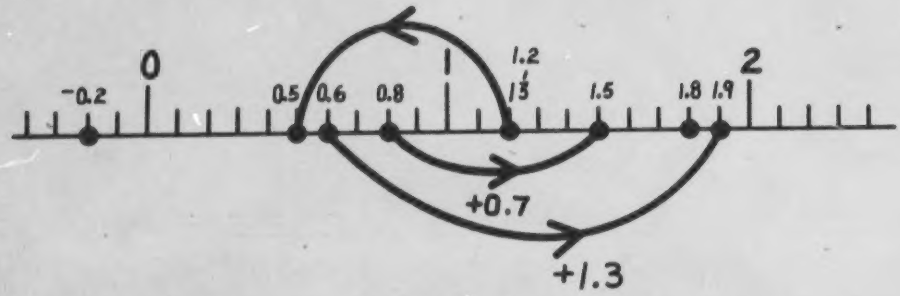
$$\begin{array}{r} 0.60 \\ +\ 0.13 \\ \hline 0.73 \end{array}$$

S: 0.13 is too small.

S: + 1.3.

T: Yes. 0.6 + 1.3 = 1.9.

Draw this green arrow.

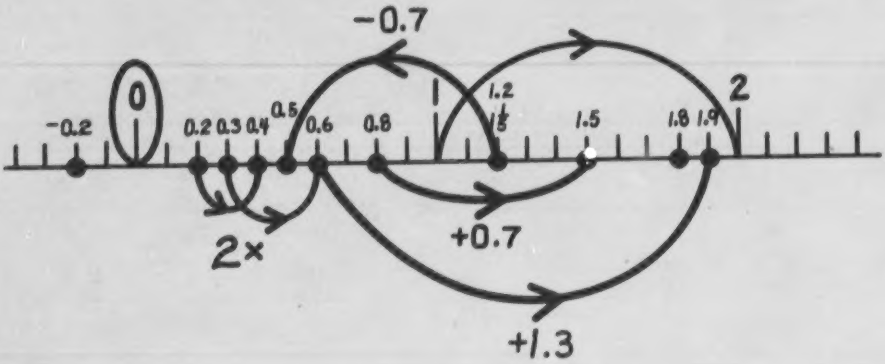

0
1
1.2
2
−0.2
0.5
0.6
0.8
1.3
1.5
1.8
1.9
+0.7
+1.3

T: What could the green arrow be for?

S: -0.7.

T: I would like someone to go to the board and draw a 2× arrow. (Use any color not yet in your picture.)

Below we have recorded (in black) a few responses that were given in our development classes.



T: Who can draw a 2× arrow starting at 0.6?

A student should draw an arrow starting at 0.6 and ending at 1.2.

T: Now, who can draw a 2× arrow ending at 1.8?

A student should draw an arrow starting at 0.9 and ending at 1.8.

Exercise 2

NOTE: In this exercise we introduce a game, the Newtonian Quiz, that will be played several times during the remainder of the year. In the discussion below, we describe a situation that occurred in one of our development classes.

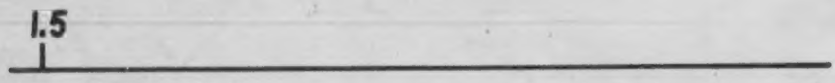
15N

Draw this line on the board.

T: I am thinking of a secret number. Would someone like to guess what it is?

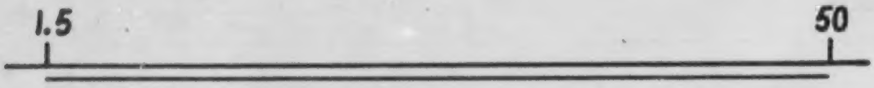
S: 1.5.

Put 1.5 on your number line.



T: My response to your guess is 50. This information tells you that the secret number is between 1.5 and 50.

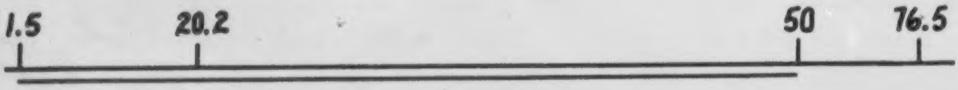
Put 50 on the number line and draw this red line segment.



T: The red line is a reminder that the secret number is between 1.5 and 50. Another guess?

S: 20.2.

T: 76.5.

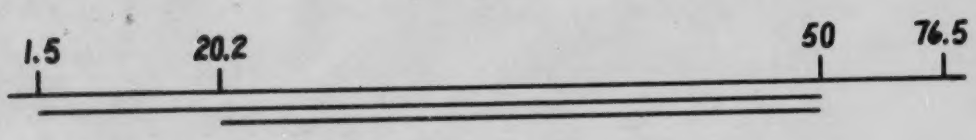


T: Now what do you know about my secret number?

S: It is between 20.2 and 76.5.

S: Yes, but it is also between 20.2 and 50.

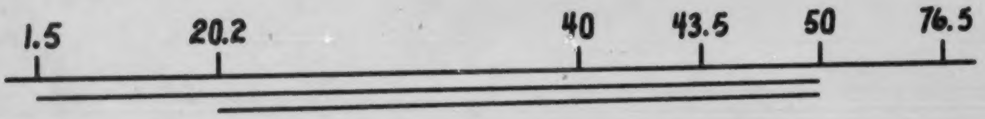
T: Very good. I will record this on the number line.


1.5
20.2
50
76.5

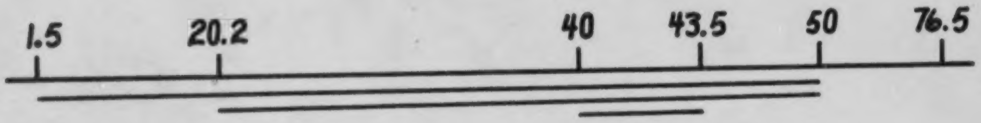
T: Do you have another guess?

S: 43.5.

T: 40.

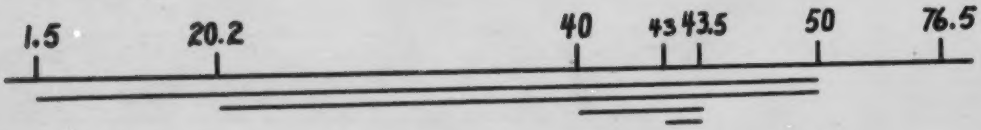

1.5
20.2
40
43.5
50
76.5

S: It's between 40 and 43.5.


1.5
20.2
40
43.5
50
76.5

S: I'll guess 43.

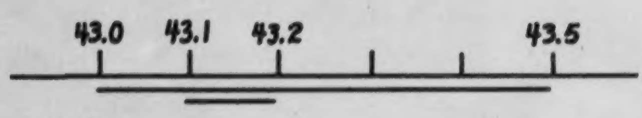
T: 43.5.


1.5
20.2
40
43
43.5
50
76.5

S: It's between 43 and 43.5. I'll guess 43.1.

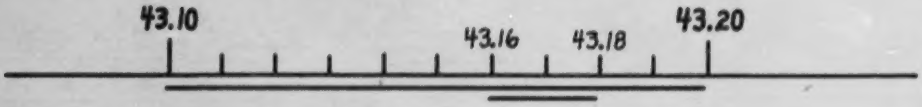
T: 43.2. Let's magnify the number line.

Draw this new part of the number line.


43.0
43.1
43.2
43.5

S: 43.16.

T: 43.18. Let's magnify one more time.

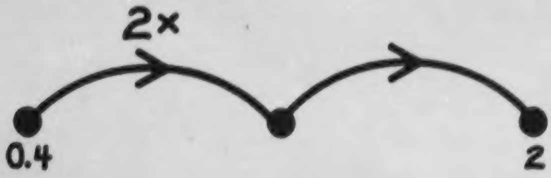

43.10
43.16
43.18
43.20

S: 43.17.

T: That's it!

Exercise 3

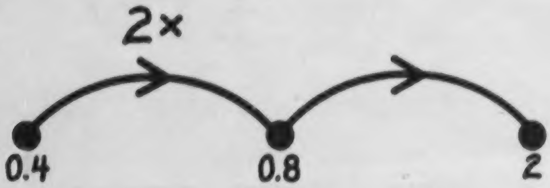
Draw this arrow picture on the board.



T: If the red arrow is for 2 ×, what number is here (point to the middle dot)?

S: 0.8.

Add this information to your picture.



T: Good. What could the blue arrow be for?

S: +0.12.

T: Let's check it.

Write the calculation on the board.

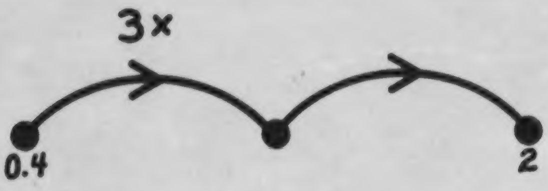
$$\begin{array}{r} 0.80 \\ +0.12 \\ \hline 0.92 \end{array}$$

T: The blue arrow can not be +0.12. What could it be?

S: +1.2.

T: Does 0.8 + 1.2 = 2? (Yes)

Change your drawing so that the red arrow is now for 3 x.



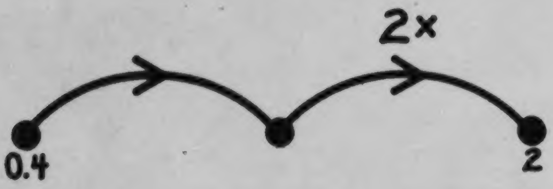
T: Suppose the red arrow is for 3 x; then what could the blue arrow be for?

S: The middle dot is for 1.2.

S: So the blue arrow could be for +0.8.

T: Does 1.2 + 0.8 = 2? (Yes)

Change your drawing as follows:



T: Suppose the blue arrow is for 2 x; then what could the red arrow be for?

S: The middle dot is for 1.

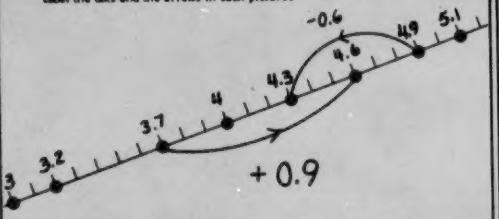
S: The red arrow could be for + 0.6, because 0.4 + 0.6 = 1.

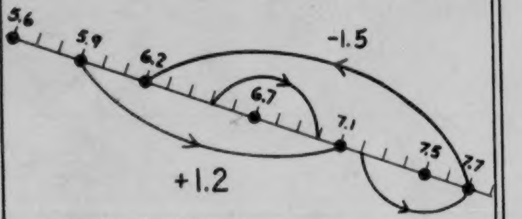
Worksheets N39, N40, and N41 are available for the rest of the lesson.

Name ____________________

N 39 *

Label the dots and the arrows in each picture.


-0.6
3
3.2
3.7
4
4.3
4.6
4.9
5.1
+ 0.9


5.6
5.9
6.2
-1.5
6.7
7.1
7.5
7.7
+1.2

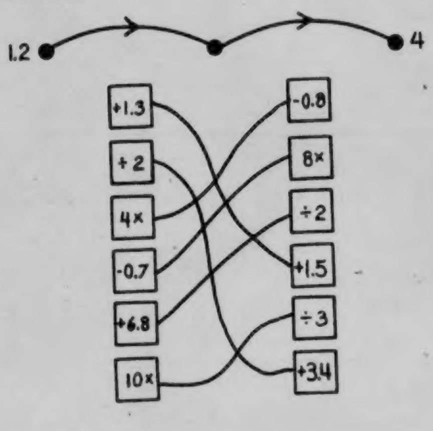
Use green to draw two +0.5 arrows.

Many other green arrows are possible.

Name ____________________

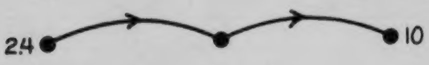
N 40 **

Pair the tags.


1.2
4
+1.3
÷ 2
4×
-0.7
+6.8
10×
-0.8
8×
÷2
+1.5
÷3
+3.4

Name ____________________

N 41 ***



Complete the chart.

→	→
2×	+5.2
÷2	+8.8
3×	+2.8
÷3	+9.2
-0.9	+8.5
-1.4	+9 or 10×
+2.6	+5 or 2×

→	→
+2.6	2×
+17.6	÷ 2
-1.4	10×
+97.6	÷ 10
+15	-7.4
+0.2	+7.4
For example: 5×	-2

3.5 Newtonian Quiz #1

Exercise 1

NOTE: In this exercise you and your class will play the Newtonian Quiz. Because, in this lesson, the purpose of playing the game is to introduce the thousandths subgraduation, you need not choose the secret number until the end of the game.* As an example of how one may control the game and very quickly "home-in" on the thousandths subgraduations, we include the dialogue of a game that was played in one of our development classes.

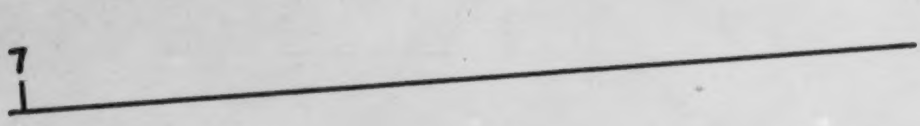
Draw this line on the board.

T: I will think of a secret number between 0 and 10.

Would someone like to guess what it is?

S: 7.

Put 7 on your number line.



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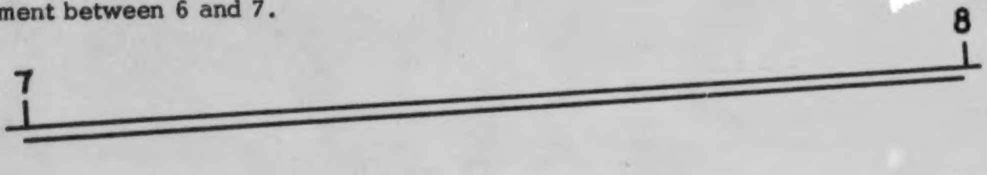
* There is a small amount of dishonesty here, since the secret number is not chosen until the end. By bending this rule, we play a shorter and more interesting game. A careful study of the dialogue should reveal to you the method of using the game to explore quickly any subgraduation you wish.

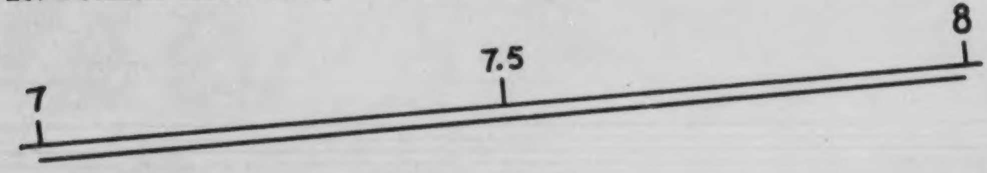
T: My response to your guess is 8. Now we know the secret number is between 7 and 8.

Put 8 on your number line and draw a red line segment. At this point the students' attention is focused on a segment of the number line between 7 and 8. Another good response would be 6; then you would focus attention on the segment between 6 and 7.



S: I'll guess 7.5.

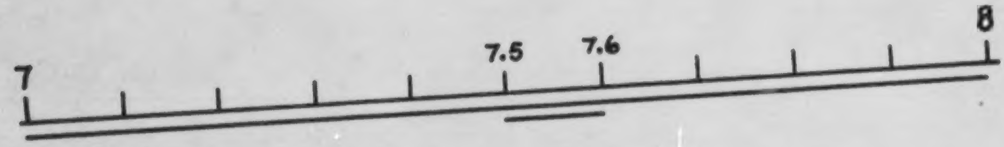
Let a student locate this point on the number line.



T: 7.6 (7.4 would also be a good response). How can we locate this number on the number line?

S: Divide the segment between 7 and 8 into ten smaller segments of the same length.

Add the tenths graduation to your drawing and let a student locate 7.6.



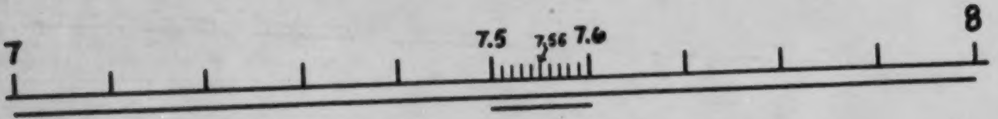
Note that through carefully selected teacher responses the search is directed to consider hundredths subgraduations.

S: 7.55.

T: Can you locate it on the number line?

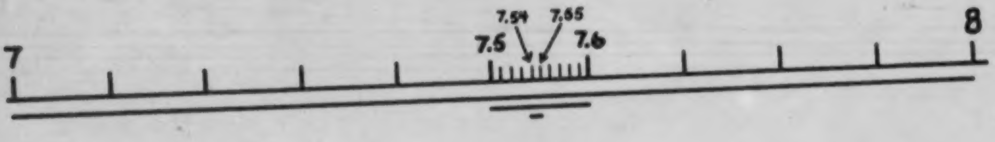
S: Divide the segment between 7.5 and 7.6 into ten smaller segments of the same length.

Add these graduations to your drawing and let a student locate 7.55.



T: Very good. My response is 7.54 (7.56 would have been just as good a response). Can you locate it on the number line?

Let a volunteer locate 7.54 on the number line.



T: Now we know the secret number is between 7.54 and 7.55. What could my secret number be?

S: 7.53.

T: Is this a good guess?

S: 7.53 is not between 7.54 and 7.55.

Let a student show that 7.53 is to the left of 7.54 on the number line.

S: 7.555.

T: Where is 7.555 on the number line?

S: It is to the right of 7.55, in the middle of 7.55 and 7.56. So, it cannot be the secret number.

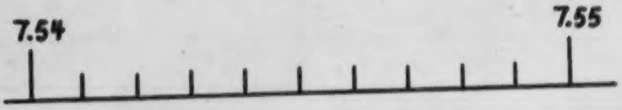
Have a student approximately locate 7.555 in your drawing.

S: 7.543.

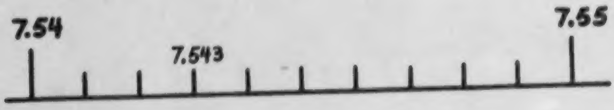
T: Can you locate it on the number line?

S: Divide the segment between 7.54 and 7.55 into ten smaller segments of the same length.

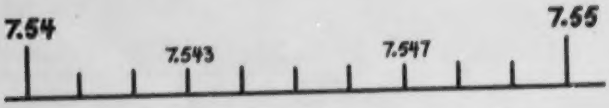
T: Good. Let's magnify our number line to show this.



Then let a student locate 7.543.

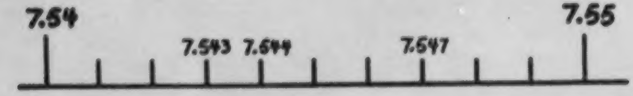


T: My response is 7.547. Can you locate it?



S: 7.544 (or any response between 7.543 and 7.547).

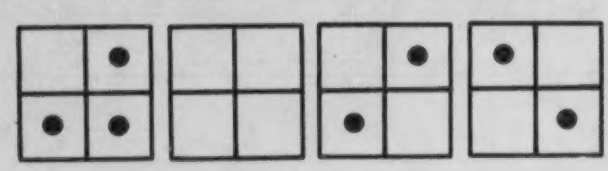
T: That's the number. Show us this number on the number line.



Exercise 2

NOTE: As an option, the hand-calculator may be used for this exercise. Since the goal of the exercise is decoding, the calculator may be used to help students "keep-up" with the calculations involved.

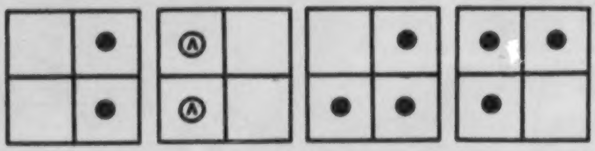
Display four Minicomputer boards.



T: What number is on the Minicomputer?

S: 7,069.

Then,



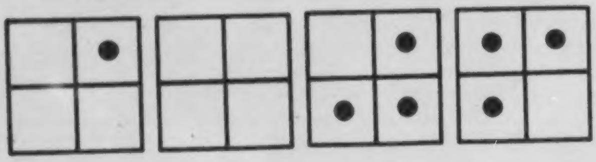
T: What number is this?

S: 4,084.

If there are students that have trouble with this problem, try asking:

T: Can you show me the same number with three fewer checkers?

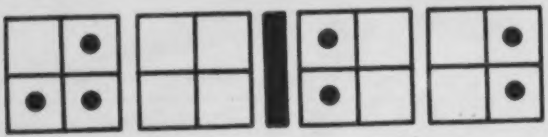
Let a student who is having difficulty with this problem come and make the plays to remove 1,000 and $^{-}1,000$.



T: Now, what number is this?

S: 4,084.

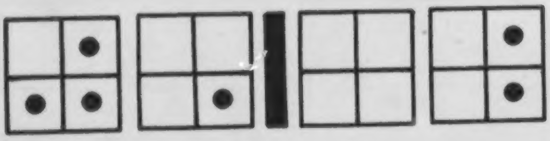
T: Let's draw the bar. What number is this?



Have the students write their answers on paper. You may discover, as you walk around observing answers, that some students are having trouble with this problem. In that case, try asking:

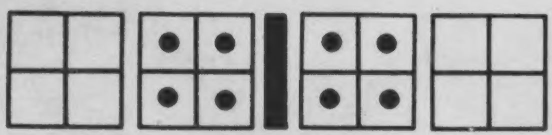
T: Can you show the same number with one less checker?

Have a student make the move.



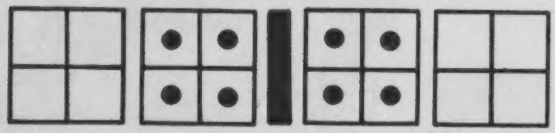
T: What number is this? (71.05)

Do a similar exercise for this configuration.



= 16.5

Return the checkers to their original configuration.



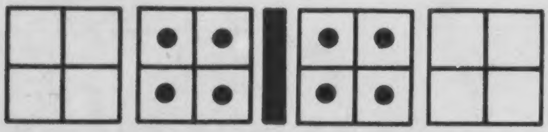
T: I will change the number by moving some checkers. Each time you must tell me if I get a larger number (and how much larger) or a smaller number (and how much smaller).

Move one checker from the 0.8-square to the 1-square. (Answer: 0.2 larger)
Move one checker from the 0.1-square to the 0.8-square. (Answer: 0.7 larger)
Move one checker from the 1-square to the 0.1-square. (Answer: 0.9 smaller)

Exercise 3

T: Now we will play a game with the Minicomputer called "Minicomputer Golf".

Put the checkers on the Minicomputer and ask the students to calculate this number.



= 16.50 = 16.5

Divide your class into three groups. One group will be Team A, another group will be Team B, and the third group (which may consist of as few as one or two students) will receive hand-calculators and will have to check progressively the results of the game.[*]

Write this information on the board.

Goal: 100

Team A: ⟶

Team B: ⟶

T: We are starting from 16.5 and the goal is 100. Students from Team A and from Team B will play in turns. Each time they can move one checker from the square it is on to any other square. The number may be increased or decreased. The first team to make us reach the goal will be the winner. Who would like to play first for Team A?

Suppose that the first player on Team A moves a checker from the 0.1-square to the 8-square. (We are describing the results of a game played in one of our development classes.)

T: Is the number on the Minicomputer larger or smaller? (Larger)

How much larger? (7.9 larger)

What number is now on the Minicomputer? (24.4)

The calculation "16.5 + 7.9 = 24.4" is to be checked by the third group with the calculators.

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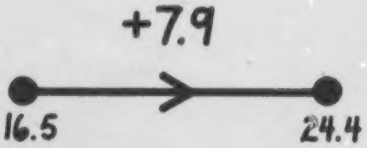
[*] If most students have calculators, the third group will not be necessary, as everyone can check the results of the game.

18 N

Record Team A's play with a red arrow.

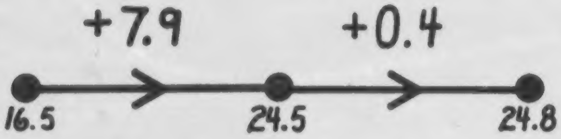


Now a player from Team B takes a turn and moves a checker from the 0.4-square to the 0.8-square.

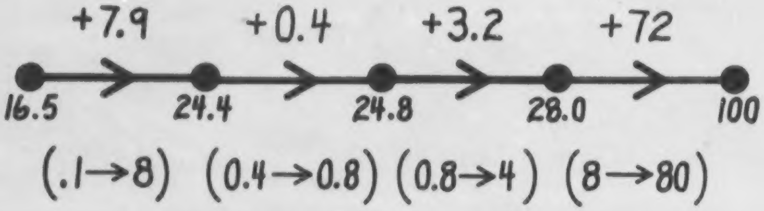
T: Is the number on the Minicomputer larger or smaller? (Larger)
How much larger? (0.4 larger)
What number is on the Minicomputer now? (24.8)

Have those with calculators check the calculation "24.4 + 0.4 = 24.8".

Record Team B's play with a blue arrow.


+7.9
+0.4
16.5
24.5
24.8

Follow similar steps for the next players, alternately from Team A and Team B. The results of our game are described by the red-blue road that follows. Team B was the winner.


+7.9
+0.4
+3.2
+72
16.5
24.4
24.8
28.0
100
(.1→8)
(0.4→0.8)
(0.8→4)
(8→80)

Worksheets N42, N43, and N44 are available.

If some of the students finish and time permits, put this problem on the board.

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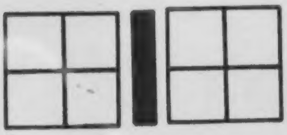
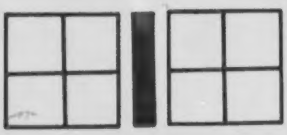
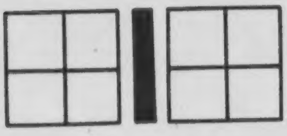
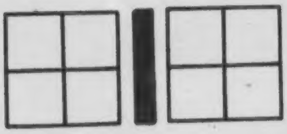
For the following numbers, use exactly two of these checkers:

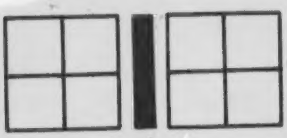
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨


= 25.8


= 75.6

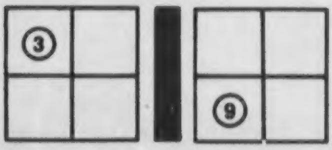

= 60.8


= 43.2

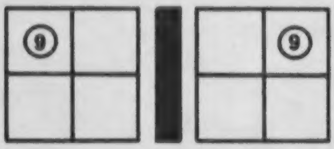

= 29.6

18 N

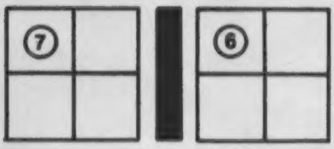
Some possible solutions are given below.



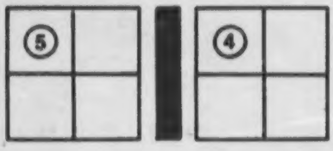
= 25.8



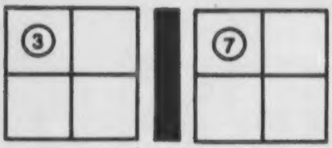
= 75.6



= 60.8



= 43.2

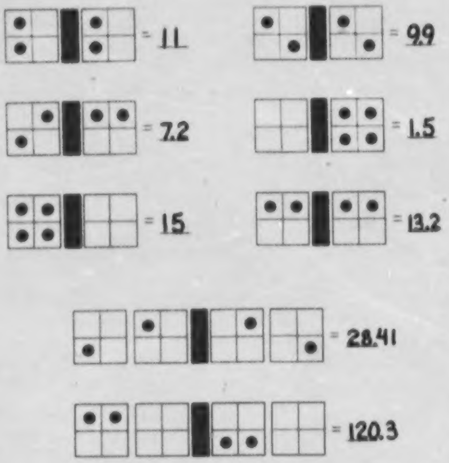


= 29.6

Name ______________________

N 42 *

Write any number you wish on the Minicomputer using exactly four positive checkers.

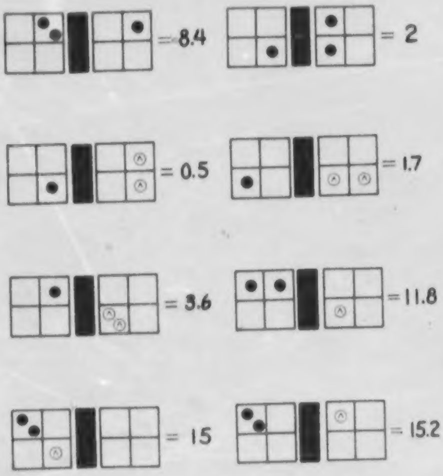

= 11
= 9.9
= 7.2
= 1.5
= 15
= 13.2
= 28.41
= 120.3

Many other solutions are possible.

Name ______________________

N 43 **

Write each of these numbers on the Minicomputer using exactly three checkers (positive or negative).


= 8.4
= 2
= 0.5
= 1.7
= 3.6
= 11.8
= 15
= 15.2

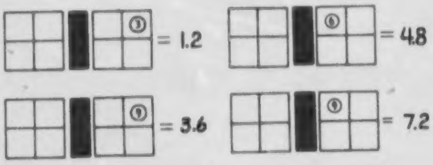
Other solutions are possible.

Name ______________________

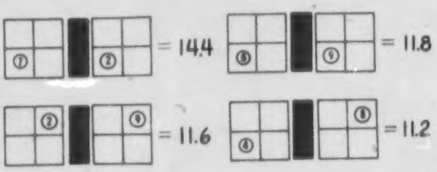
N 44 ***

Write each of these numbers on the Minicomputer using exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



For the following numbers, use exactly two of the checkers above.


= 14.4
= 11.8
= 11.6
= 11.2

Other solutions are possible.

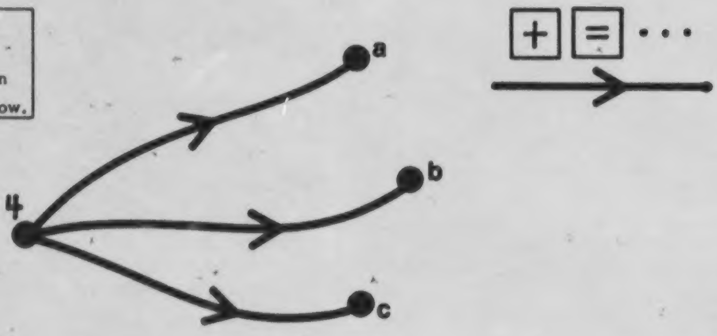
3.6 Multiples of Decimal Numbers

Exercise 1

Distribute as many hand-calculators as you have available.

Draw this arrow picture on the board.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



T: The red arrow is for $\boxed{+}$ $\boxed{=}$ ..., as many times as you want. We will start with 4. (Point to a.) What number could be here?

Let several students respond and label a and b with two answers that you receive. Then point to c.

T: What is the largest number smaller than 1,000 that could be here? (Answer: 996)

Label c "996".

T: How many times would you have to press $\boxed{=}$ to get 1,000 on the display?

S: 250 times.

T: How did you get 250?

S: 250 × 4 = 1,000.

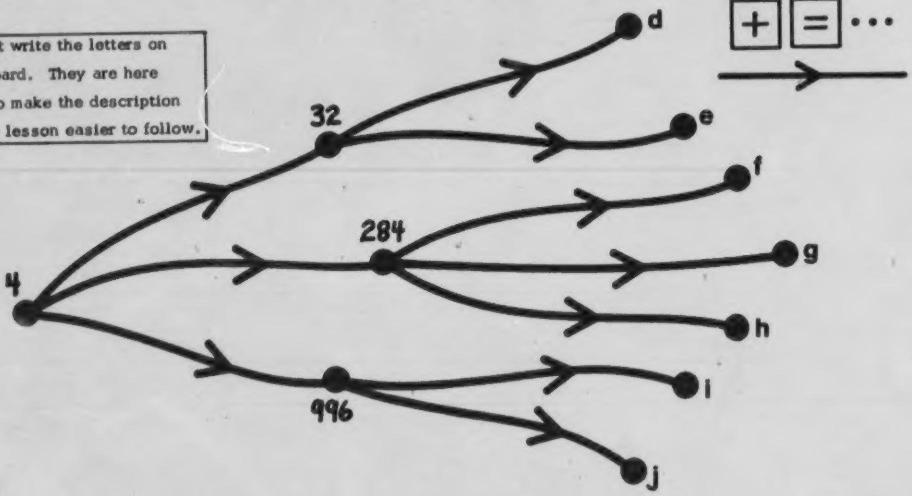
T: Very good. Now how many times would you press $\boxed{=}$ to get 996?

S: 249 times, because 996 is just four less than 1,000.

So, 249 × 4 = 996.

Add these arrows to your drawing. For the sake of this discussion, we have replaced "a" and "b" with two possible labels.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



T: Now we begin with 32. (Point to d.) Put 32 on the display of your calculator and press $\boxed{+}$ $\boxed{=}$ ..., as many times as you like. What number could this be?

Let several students suggest answers and label d and e with two of them.

T: Starting with 284 on your display, press $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ exactly ten times. What number is on the display? (Answer: 2,840)

Label f "2,840".

T: If we start with 284 and press $\boxed{=}$ one hundred times, what number would we get? (Answer: 28,400)

Label g "28,400".

T: If we again start with 284 and press $\boxed{=}$ one thousand times? (Answer: 284,000)

Label h "284,000".

T: Now start with 996. (Point to i.) What number could this be?

S: 9,960 (for example).

T: How many times 996 is 9,960?

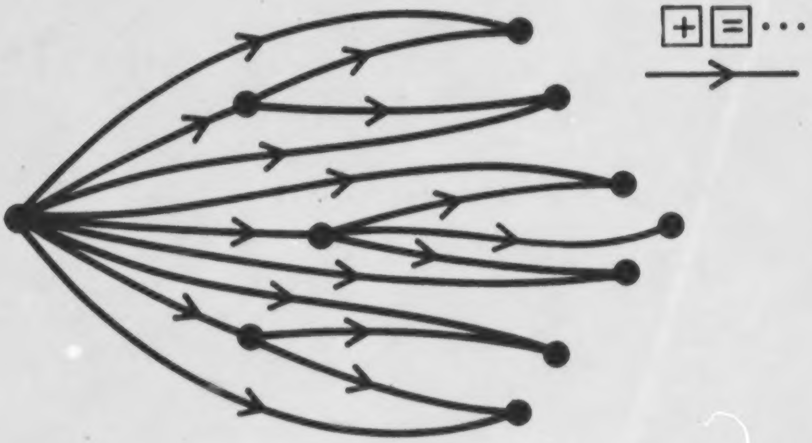
S: 10 times, because 10 × 996 = 9,960.

T: If I pressed $\boxed{=}$ one hundred times, what number could go here (point to j)? (Answer: 99,600)

Erase the numerals from the picture.

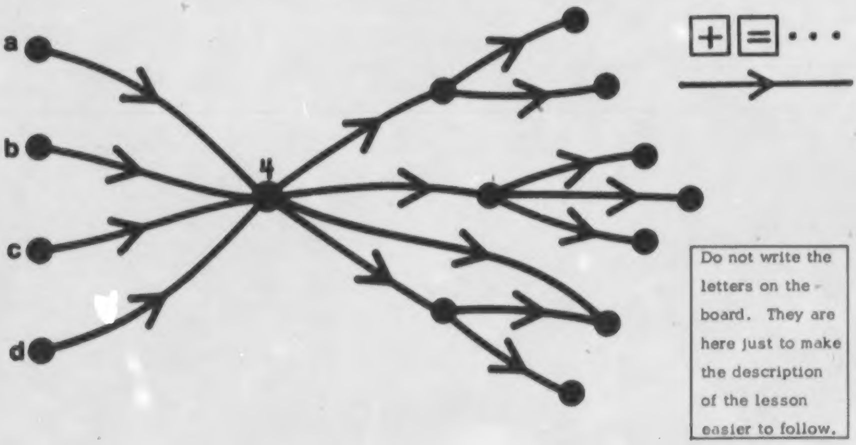
T: There are some red arrows that could be added to our picture. Who would like to trace one of these arrows in our picture?

Let a student come to the board and trace an arrow. There are many possibilities. The picture below shows many arrows that might be suggested by your class.



Trace a few more arrows if there is student interest.

Expand your arrow picture by adding these arrows on the left.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: What number could be here (point to a)?

1 and 2 are likely responses to this question. In any case, label a and b with two student responses. Suppose a student suggests 0.5 for c.

T: Let's check 0.5. Put "0.5" on the display and press $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ and so on. Does "4" appear on the display? How many times 0.5 is 4?

S: 8 times, because 8 x 0.5 = 4.

S: 0.25 works.

T: Put "0.25" on the display and press $\boxed{+}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$. What number is on the display? (Answer: 1)

T: How many times 0.25 is 4? (Answer: 16)

Some students may have difficulty with this question. For their benefit, ask:

T: If I press $\boxed{=}$ four more times, what number will be on the display? (Answer: 2)

And if I press $\boxed{=}$ four more times? (Answer: 3)

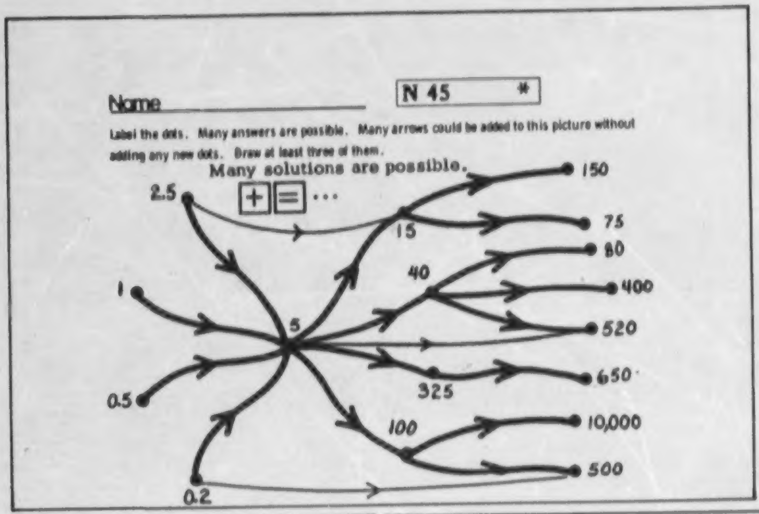
And if I press $\boxed{=}$ four more times? (Answer: 4)

How many times 0.25 is 4?

S: 16 times.

Continue in this manner with one or two more examples.

Worksheets N45, N46, and N47 are available for the remainder of the lesson.

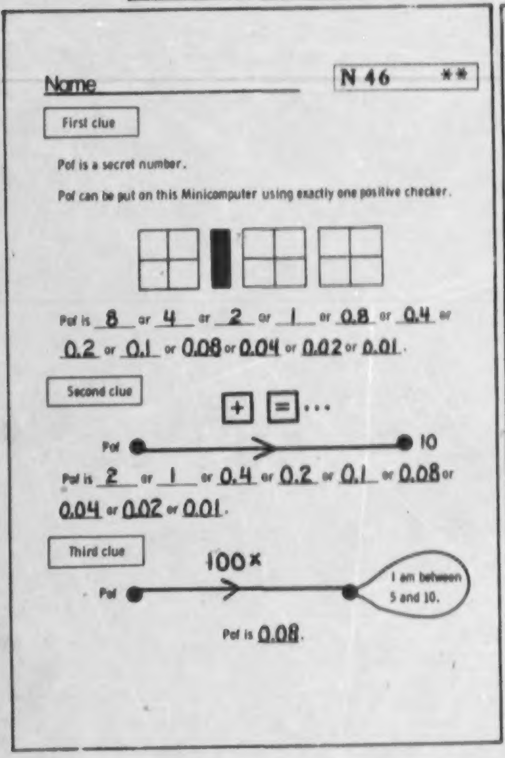


Name________________ **N 45** *

Label the dots. Many answers are possible. Many arrows could be added to this picture without adding any new dots. Draw at least three of them.

Many solutions are possible.

+
=
...
2.5
1
0.5
0.2
5
15
40
325
100
150
75
80
400
520
650
10,000
500



Name________________ **N 46** **

First clue

Pof is a secret number.

Pof can be put on this Minicomputer using exactly one positive checker.

Pof is 8 or 4 or 2 or 1 or 0.8 or 0.4 or 0.2 or 0.1 or 0.08 or 0.04 or 0.02 or 0.01.

Second clue

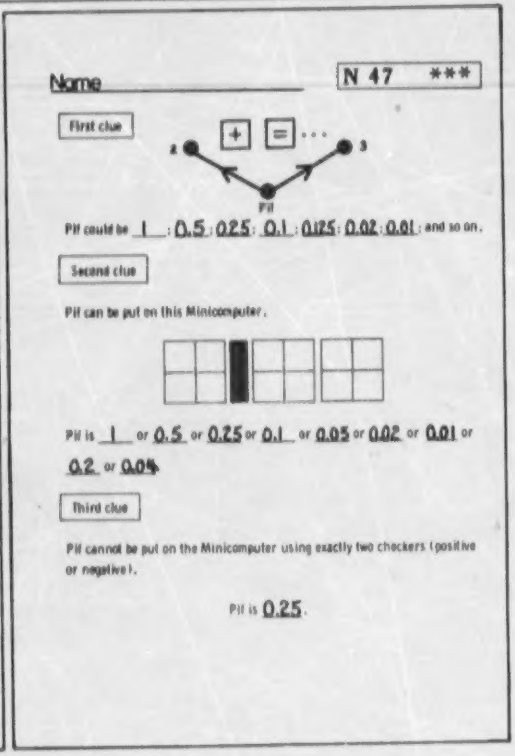
+
=
...
Pof
10

Pof is 2 or 1 or 0.4 or 0.2 or 0.1 or 0.08 or 0.04 or 0.02 or 0.01.

Third clue

100x
Pof
I am between 5 and 10.

Pof is 0.08.



Name________________ **N 47** ***

First clue

Pil could be 1 ; 0.5 ; 0.25 ; 0.1 ; 0.125 ; 0.02 ; 0.01 ; and so on.

Second clue

Pil can be put on this Minicomputer.

Pil is 1 or 0.5 or 0.25 or 0.1 or 0.05 or 0.02 or 0.01 or 0.2 or 0.04

Third clue

Pil cannot be put on the Minicomputer using exactly two checkers (positive or negative).

Pil is 0.25.

3.7 Newtonian Quiz #2

Exercise 1

NOTE: In this exercise you and your class will play the Newtonian Quiz. Since the purpose of playing the game is to explore the ten-thousandths subgraduation, you need not choose the secret number until the end of the game.* As an example of how one may control the game and very quickly "home-in" on the ten-thousandths subgraduations, we include the dialogue of a game that was played in one of our development classes.

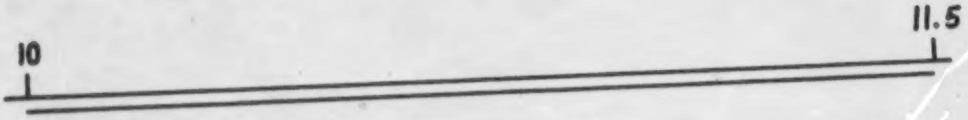
Draw this line.

T: I am thinking of a secret number and would like someone to guess what it is.

S: 10.

T: My response is 11.5.

Put 10 and 11.5 on your number line and draw the red line segment.



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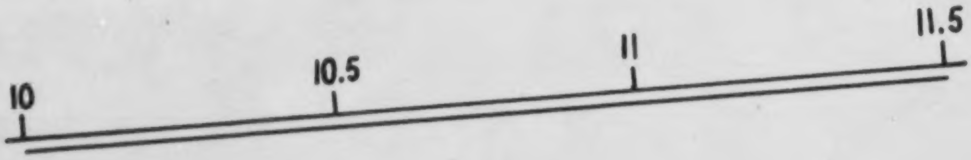
* There is a small amount of dishonesty here, since the secret number is not chosen until the end. By bending this rule, we play a shorter and more interesting game. A careful study of the dialogue should reveal to you the method of using the game to explore quickly any subgraduation you wish.

T: My response tells you that the secret number is between 10 and 11.5. For example, 11 is between 10 and 11.5. How can we locate 11 on the number line?

S: Divide the segment from 10 to 11.5 into thirds.

Let a student make the divisions and label the number line.



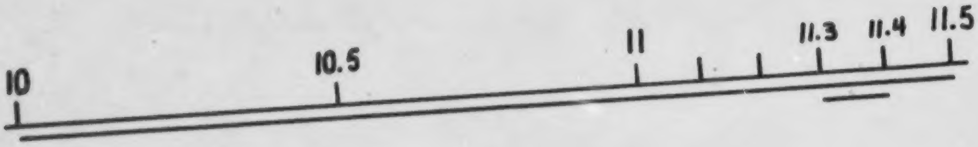
T: Now you know the secret number is between 10 and 11.5. What is your next guess?

S: 11.3.

T: My response is 11.4 (your response should be 0.1 more or less than the guess). How can we locate 11.3 and 11.4 on the number line?

S: Divide the segment between 11 and 11.5 into five smaller segments of the same length.

Add these five subgraduations to your drawing and let someone locate 11.3 and 11.4.



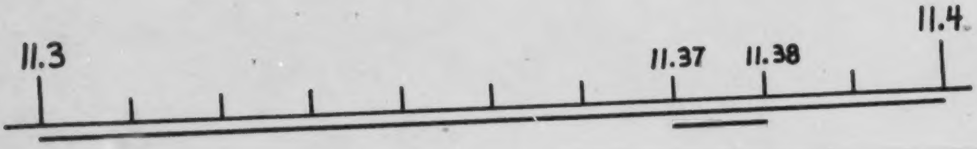
S: 11.38.

T: 11.37 (this time your response should be 0.01 more or less than the guess). How can we locate these numbers on the number line?

S: Divide the segment between 11.3 and 11.4 into ten smaller segments of the same length.

T: Very good. First, let us magnify part of the number line.

Draw this "magnified" number line and let a student locate 11.38 and 11.37.



T: Another guess?

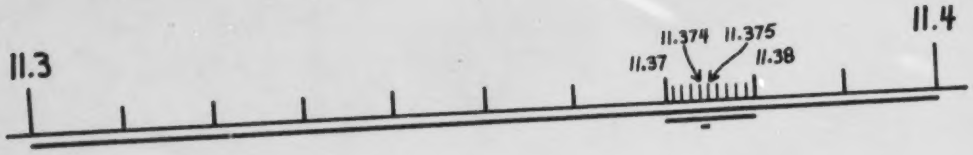
S: 11.374.

T: How can we locate 11.374 on the number line?

S: Divide the segment between 11.37 and 11.38 into ten smaller segments of the same length.

T: Very good. 11.374 is not the secret number. Instead I respond 11.375.

Make the subgraduation and let someone locate 11.375 and 11.374.

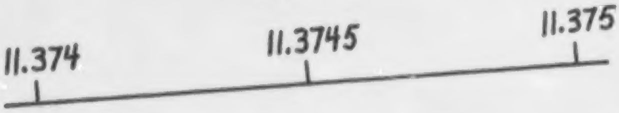


S: 11.3745.

T: That is the secret number. (Accept as the secret number any number between 11.374 and 11.375.)

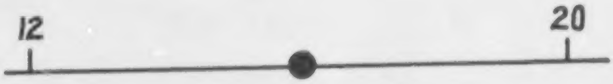
Who can locate this number on a "magnified" number line?

Draw this number line on the board and let someone locate 11.3745. It is likely that a student will indicate that 11.3745 is the midpoint of the segment that you have drawn.



Exercise 2

Draw this picture.



T: I have drawn a portion of the number line and have located 12 and 20. The dot is for the number that is half-way between 12 and 20; it is the midpoint of 12 and 20. What number is it?

S: 16.

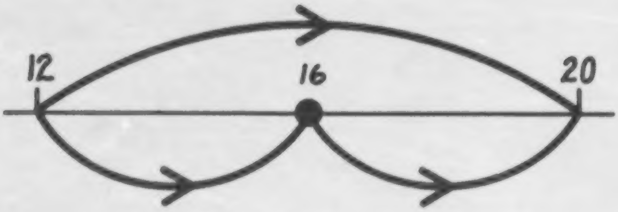
T: Good. How can we check that 16 is the midpoint?

S: 16 is four more than 12 and 20 is four more than 16.

T: Very good. Here is a way to picture the situation.

29N

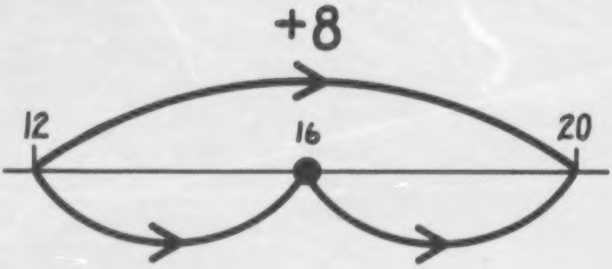
Add these arrows to your picture.



T: What could the red arrow be for?

S: + 8.

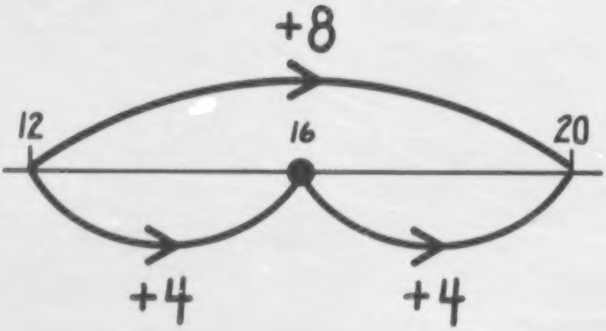
Label the red arrow "+ 8".



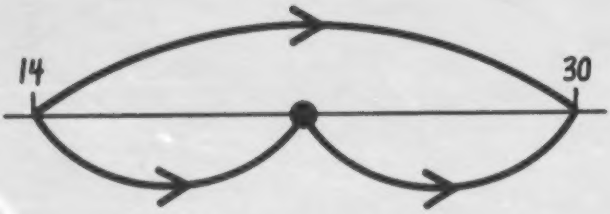
T: What could the blue arrows be for?

S: + 4.

Label the blue arrows "+ 4".



Erase the numerals and arrow labels. Label the end points "14" and "30" as shown.



T: What is the midpoint of 14 and 30?

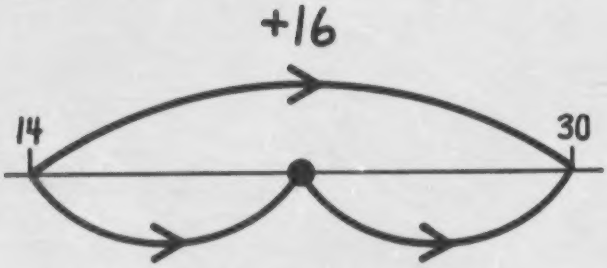
Write it on your paper.

For students that have difficulty with this situation, encourage them to first consider the arrows.

T: What could the red arrow be for?

S: + 16.

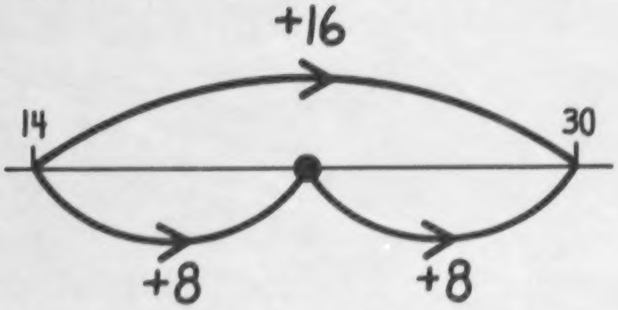
Label the red arrow.



T: What could the blue arrows be for?

S: + 8.

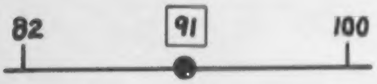
Label the blue arrows.

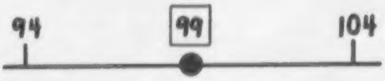


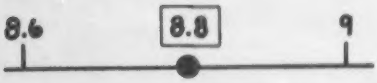
T: Now what is the midpoint of 14 and 30?

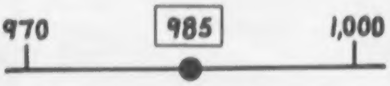
S: 22.

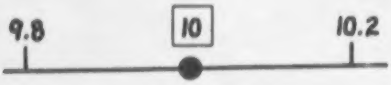
Label the midpoint "22". Erase the numerals and arrow labels, but keep the arrows. Repeat the exercise several times more. Below are some suggestions for problems with answers in the boxes. Feel free to adjust the problems to the numerical level of your class.





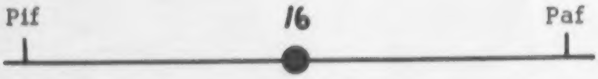



970
985
1,000



Exercise 3

Draw this picture.



T: Two secret numbers, Pif and Paf, have a midpoint of 16. What could Pif and Paf be?

Ask for several answers to this problem.

Record the answers in a table similar to the one below.

Pif	Paf
12	20
13	19
14	18
15	17

T: Do you notice any pattern in our table?

You will probably get several good answers to this question. Accept any reasonable answer as correct. However, there is one pattern that may be useful a little later.

S: If you add Pif and Paf together, you get 32. For example, 12 + 20 =32 and 14 + 18 = 32.

S: 32 is two times 16, the midpoint.

If this pattern is not suggested by the class, mention it yourself.

Then extend your table as shown below. (Do not write the answers that are boxed.)

Pif	Paf
0	$\boxed{32}$
$\boxed{7}$	25
$^{-}2$	$\boxed{34}$
$\boxed{14.5}$	17.5

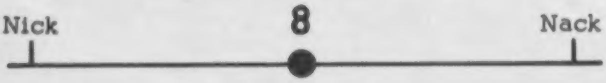
T: If Pif is 0, then Paf is ...?

S: 32.

Continue in this manner until the table is complete.

Exercise 4

Draw this picture on the board.



T: We are going to solve a detective story about Nick and Nack.

First clue

8 is the midpoint of Nick and Nack. What could Nick and Nack be?

Let students suggest several numbers that Nick and Nack could be and record their suggestions.

Nick	Nack
4	12
9	7
16	0

T: If Nick is 5, then Nack is ...?

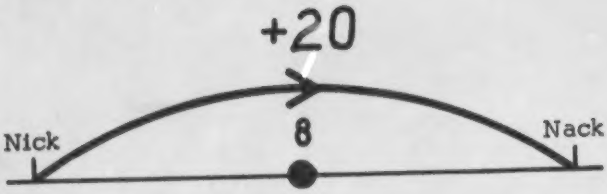
S: 11.

T: If Nick is $^{-}1$, then Nack is ...?

S: 17.

Second clue

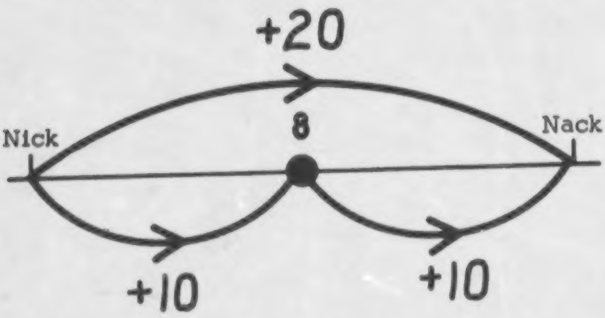
Add this arrow to your picture.



T: Nack is exactly 20 larger than Nick. With this clue you can find Nick and Nack. Write what Nick and Nack are on your paper.

The class should conclude that Nick is ⁻2 and Nack is 18.

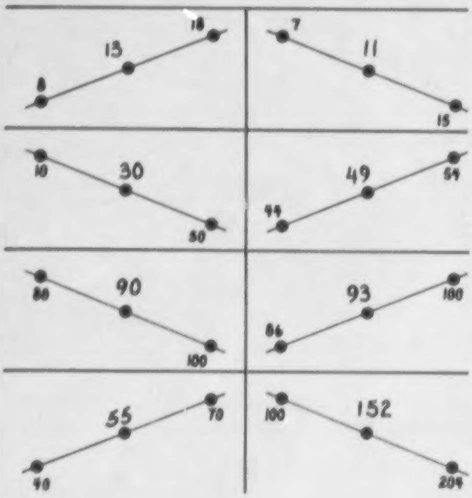
If there are some students who have difficulty finding Nick and Nack, add these blue arrows to your picture and ask them to label the arrows.



Worksheets N48, N49, N50, and N51 are available.

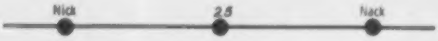
Name ______________________ N 48 *

Label each midpoint.


8
13
18
7
11
15
10
30
50
44
49
54
80
90
100
86
93
100
40
55
70
100
152
204

Name ______________________ N 49 **

25 is the midpoint of Nick and Nack.


Nick
25
Nack

Complete:

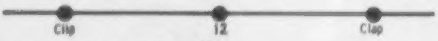
Nick	Nack
23	27
21	29
20	30
18	32
10	40
9	41
0	50
1	49
¯1	51

Nick	Nack
19	31
17	33
15	35
13	37
5	45
3	47
¯5	55
24.5	25.5
23.5	26.5

Name ______________________ N 50 ***

First clue

12 is the midpoint of Clip and Clap.


Clip
12
Clap

If Clip were 3, Clap would be 21.

If Clip were ¯2, Clap would be 26.

If Clap were 20, Clip would be 4.

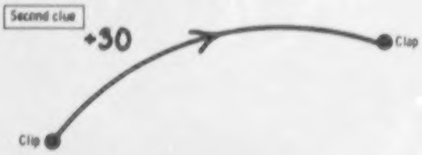
If Clap were 30, Clip would be ¯6.

If Clip were ¯10, Clap would be 34.

If Clip were 9.5, Clap would be 14.5.

If Clap were 15.3, Clip would be 8.7.

Second clue

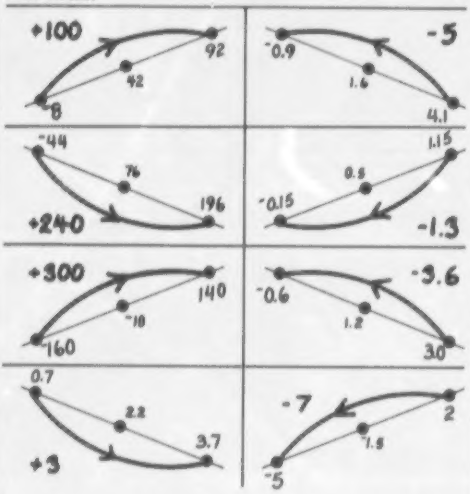

+30
Clap
Clip

Clip is ¯3. Clap is 27.

Name ______________________ N 51 ****

In each picture the in-between dot is the midpoint.

Label the dots.


+100
92
42
¯8
-5
¯0.9
1.6
4.1
¯44
76
196
+240
1.15
0.5
¯0.15
-1.3
+300
140
¯10
¯160
-3.6
¯0.6
1.2
3.0
0.7
2.2
3.7
+3
-7
2
¯1.5
¯5

4

THE TELEPHONE GAME

A short reminder of the Telephone Company's rule for making phone calls in the World of Numbers as introduced in the workbook THE TELEPHONE GAME* reintroduces an interesting situation in which to explore some properties of the integers. The situation makes use of the symmetric relation:

> two integers are related if and only if one is the double of the other or one is ten more than the other.

In the context of working with this relation, a great deal of mental arithmetic and exploration of the integers is accomplished.

CAPSULE LESSON SUMMARIES

Section 4.1:	Review the rule for making telephone calls devised for the integers in THE TELEPHONE GAME and examine some consequences of this rule. Using this rule, build cord roads (in this context, telephone communication lines) and interesting patterns of connection between the integers. (Worksheets N52, N53, N54)
Section 4.2:	Continue to develop the exercises of Section 4.1. Extend the idea of building cord roads (telephone communication lines) to longer, more challenging situations and determine the length of these roads. Examine integers for which there is no road between them. (Worksheets N55, N56)

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* THE TELEPHONE GAME is one of the workbooks used in CSMP Mathematics for the Upper Primary Grades, Part II and for the Intermediate Grades, Entry Module I.

4.1 Telephone Game #1

Exercise 1

T: Do you remember when we played the Telephone Game? Do you remember the rule for making telephone calls?

TWO INTEGERS CAN CALL EACH
OTHER BY PHONE

if and only if

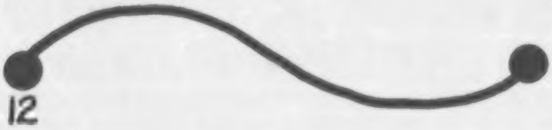
ONE OF THEM IS THE DOUBLE OF
THE OTHER

or

ONE OF THEM IS 10 MORE THAN
THE OTHER

Remind your students that an integer is a whole number or the opposite of a whole number. For example, 3; 75; $^{-}12$; 0; $^{-}151$ and so on are integers and $2\frac{1}{2}$; 3.1; $^{-}5\frac{1}{4}$ and so on are not integers.

Draw the following picture on the board.



T: Which numbers can 12 call?

If a student answers, for instance, 22 (because $22 = 12 + 10$) label the right dot "22" and ask:

T: What other numbers can 12 call?

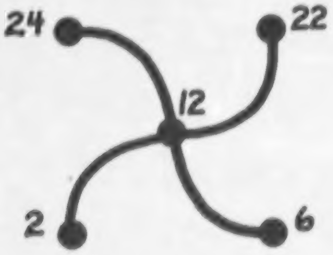
The students will probably answer:

S: 2, because $2 + 10 = 12$ (or $12 - 10 = 2$);

24, because $2 \times 12 = 24$ (or $\frac{1}{2} \times 24 = 12$);

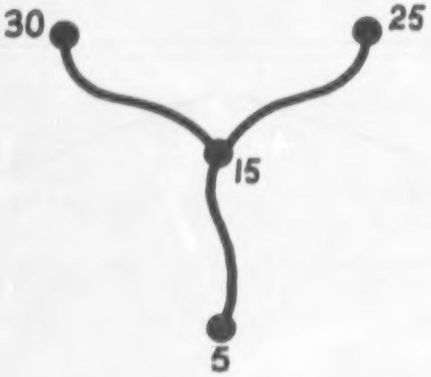
6, because $2 \times 6 = 12$ (or $\frac{1}{2} \times 12 = 6$).

Complete the picture on the board.



Erase the numbers, label the central dot "15", and ask similar questions. Probably the students will say that 15 can call 25, 5, 30 and 7.5.

T: 7.5 is a good answer, but in this game we play only with the integers and 7.5 is not an integer. So 15 can only call his three friends: 25, 5 and 30.



Replace "15" by "4" and ask similar questions. (Answers: 8; 2; 14; $^{-}6$)

T: Using any numbers you like, find some numbers that can call exactly three number friends and some numbers that can call exactly four number friends.

Give the students a minute or two to explore this situation before letting volunteers tell you the numbers that they found. Some students may suggest that odd numbers can call exactly three number friends and that even numbers can call exactly four number friends. Should this happen, ask the students to find the number friends that 0, 10, 20, $^-10$, or $^-20$ can call.

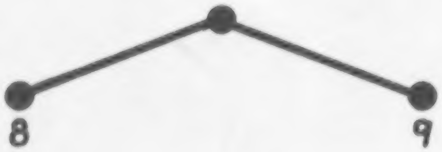
T: Give me two numbers which cannot call each other by phone.

Encourage the children to give several examples and in each case, ask them to explain why the numbers concerned cannot call each other. For instance,

S: 10 and 11 because 11 is 1 more (and not 10 more) than 10. Also, neither of them is the double of the other.

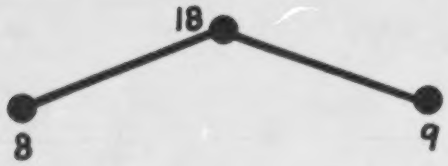
Exercise 2

Draw this picture.

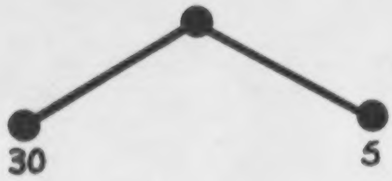


T: 8 cannot call 9, but 8 is helped by a friend. Who could the friend be?

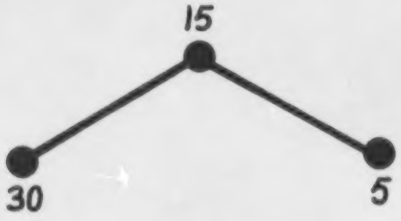
Let the students solve this problem at their seats. Then have a student come to the board and label the dot.

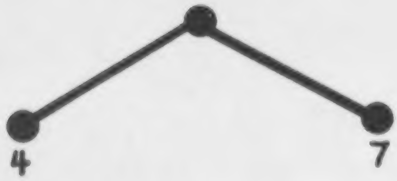


Continue in this manner with the following problems.

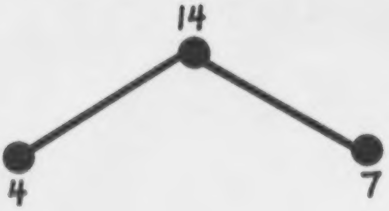


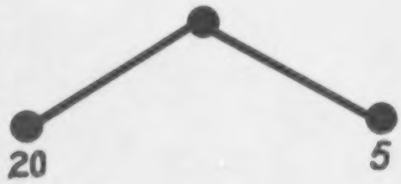
Solution :



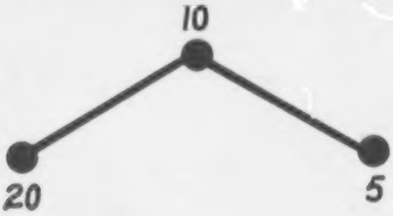


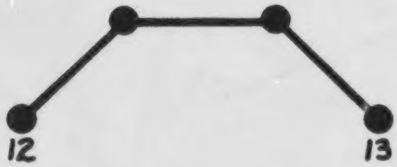
Solution :



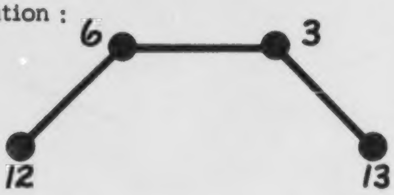


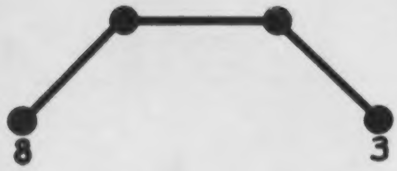
Solution :



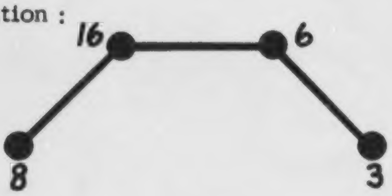


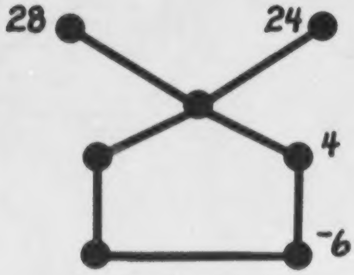
Solution :



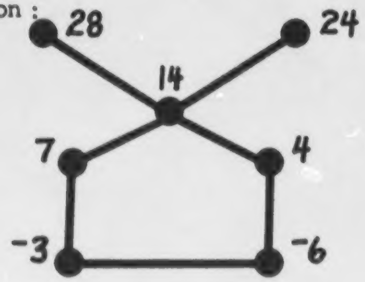


Solution :





Solution :


28
24
14
7
4
-3
-6

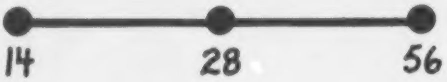
Exercise 3

T: In the World of Numbers, it costs 10¢ to make a telephone call. Who can tell me one number 14 can call for 20¢?

Suppose a student answers:

S: 56, because 14 can call 28 and 28 can call 56.

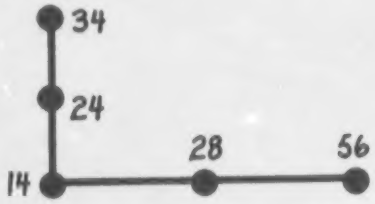
Then draw a picture to show this information.



T: What is another number that 14 can call for 20¢?

S: 34, because 14 can call 24 and 24 can call 34.

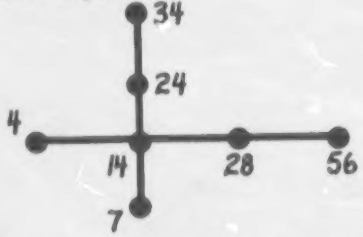
Add this information to your drawing.


34
24
28
56
14

T: Let's see if we can find all the numbers that 14 can call for 20¢. First, who can tell me the numbers that 14 can call for 10¢?

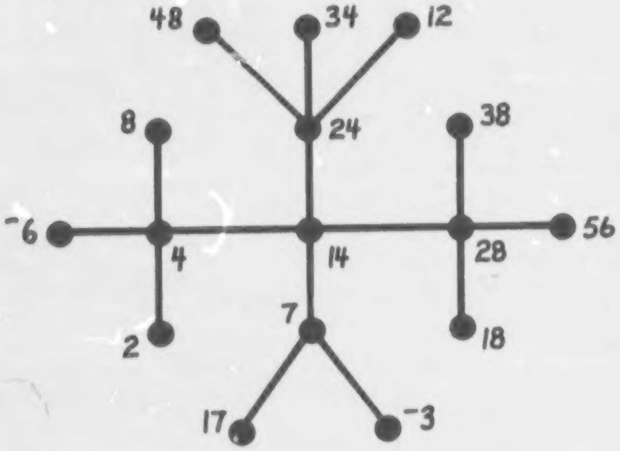
S: 28, 24, 4 and 7.

Add this information to your drawing.


34
24
4
14
28
56
7

T: Now, let's finish the drawing by finding all the numbers that 14 can call for 20¢.

As students suggest numbers that 14 can call, add them to your picture. The completed picture should look similar to the one below.


48
34
12
8
24
38
⁻6
4
14
28
56
7
2
18
17
⁻3

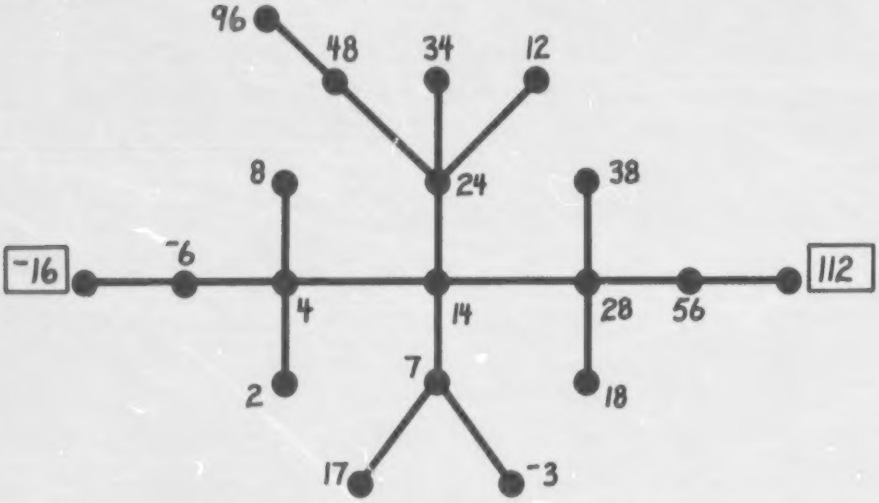
T: Which are the largest and the smallest numbers 14 can call for 30¢?

Let the class have a few minutes to consider the situation.

S: 96 is the largest, because 48 x 2 = 96.

T: I agree that 14 can call 96 for 30¢. But is 96 the largest number 14 can call for 30¢?

Add 96 to your drawing at the appropriate place and go on trying to find other numbers that 14 can call for 30¢ in order to determine the largest and smallest. (The largest and smallest numbers 14 can call for 30¢ are in the boxes.)



T: Can anyone find a number larger than 96?

S: Yes, 112.

Continue in this manner until the largest and smallest numbers 14 can call for 30¢ are determined.

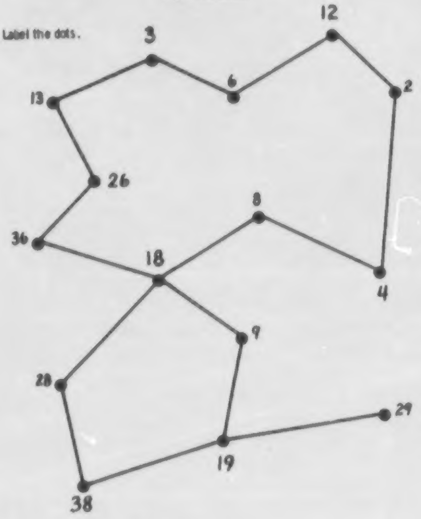
Worksheets N52, N53, and N54 are available for the rest of the lesson.

Name ____________________

N 52 *

Telephone Game

Label the dots.



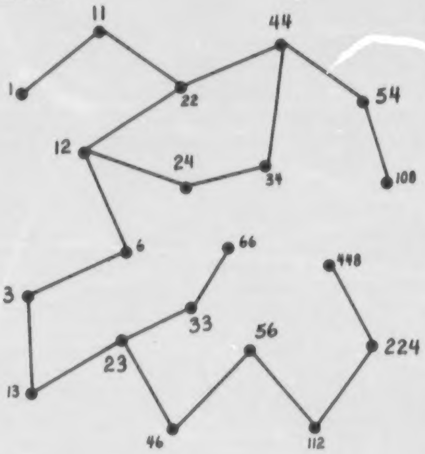
12
3
13
6
2
26
8
36
18
4
9
28
29
19
38

Name ____________________

N 53 **

Telephone Game

Label the dots.



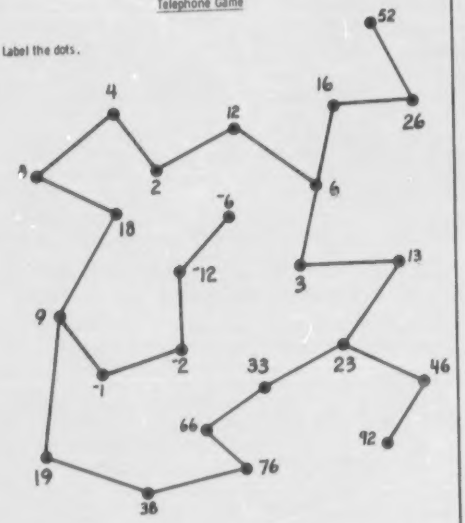
11
44
1
22
54
12
24
34
108
66
6
448
3
33
56
23
224
13
46
112

Name ____________________

N 54 ***

Telephone Game

Label the dots.



52
16
26
4
12
8
2
6
-6
18
-12
3
13
9
-2
-1
33
23
46
66
92
19
76
38

4.2 Telephone Game #2

Exercise 1

T: Do you remember when we played the telephone game a few weeks ago? Do you remember the rule for making telephone calls?

TWO INTEGERS CAN CALL EACH OTHER BY PHONE

if and only if

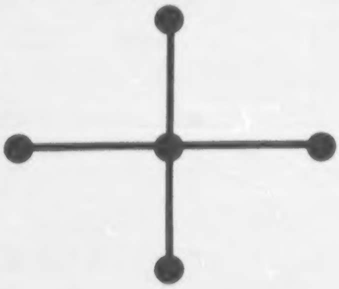
ONE OF THEM IS THE DOUBLE OF THE OTHER

or

ONE OF THEM IS 10 MORE THAN THE OTHER

Remind your class that an integer is a whole number or the opposite of a whole number (i.e., a negative integer or zero). For example, 5; 8; 6; $^{-}57$; and so on are integers, but $\frac{1}{2}$; $\frac{2}{3}$; $3\frac{1}{4}$; 2.07; 32.5; 0.52 and so on are not integers.

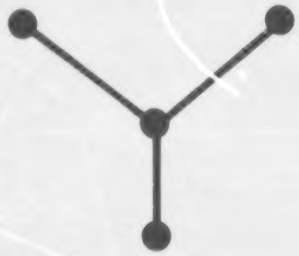
Draw the following picture on the board.



T: Can you think of a number which could call four of its friends?

Let the students give many answers, for example, 40; 8; 2; 18; etc.

Draw the following picture on the board.



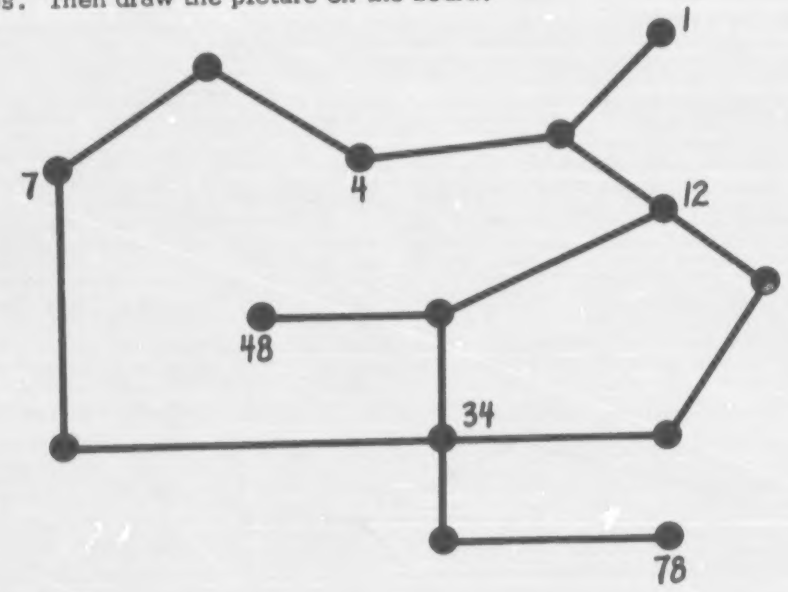
T: Can you think of a number which can only call three of its friends?

Let the class give many answers, for example, 20; 10; 5; 7; 35; etc.

Exercise 2

T: This is a detective story about Zip.

Distribute Worksheet N55 and let the class work on the first clue for a few minutes. Then draw the picture on the board.

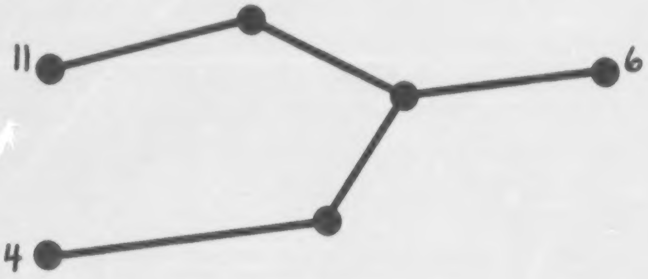


T: Zip is in this picture. Its dot is not labeled. Who can come to the board and label a dot?

Let students come to the board and label the dots. They should conclude that Zip is 2, 14, 17, 22, 24, 44, or 68.

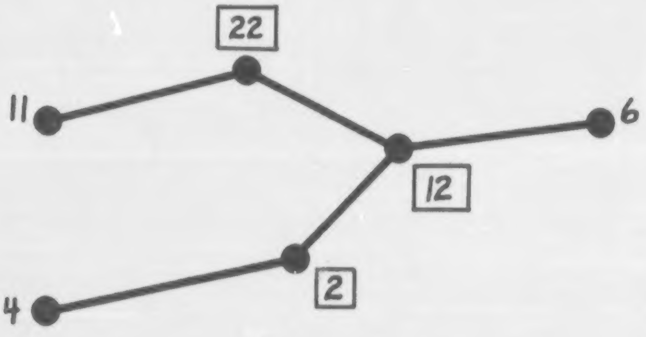
Give students time to work on the second clue, Worksheet N56 that is on the back of N55. It might be necessary to remind them that a phone call costs 10¢. After a few minutes, bring the class together to solve the second clue. A picture showing all the numbers that 8 can call for 20¢ is at the end of this lesson. The class should conclude that Zip is 14 or 2.

Draw this picture on the board.



T: The third clue is that Zip is in this picture but its dot is not labeled. Who would like to come to the board and label a dot?

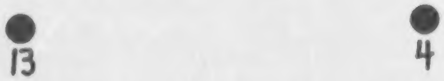
Let students come to the board and label the dots. The completed picture should look like the picture below. The answers are in the boxes.



The class should conclude that Zip is 2, since only 2 appears in the first and second clues.

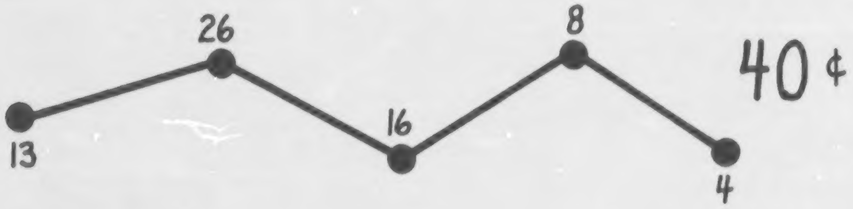
Exercise 3

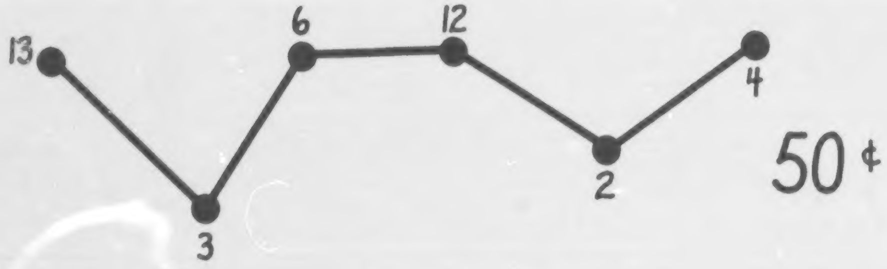
Draw this picture.



T: 13 cannot call 4 directly. But with the help of some friends 13 can talk to 4. Who could the friends be? Remember in the World of Numbers it costs 10¢ to make a phone call. Try to find the least expensive way to call.

Let the class have a short while to work on this problem. Since there are many possible solutions, let a few students put their solutions on the board. Two possible solutions are given below.



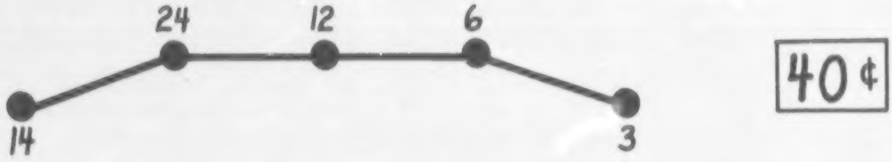


Now write these problems on the board and let the class solve them in a manner as above.

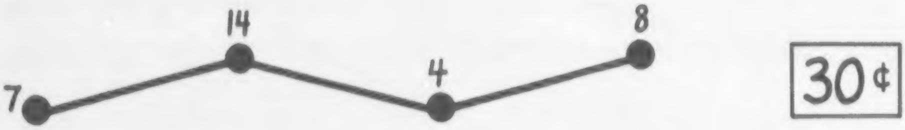
*	**	***
14 to 3	16 to 17	$^{-}13$ to 13
7 to 8	1 to $^{-}1$	15 to 16
	$^{-}12$ to 12	

As the students are working, walk around and monitor student progress. If time permits, put a few student solutions on the board and challenge other students to find a cheaper solution if possible. Some possible answers to the problems follow. In each case the solution is the cheapest one possible.

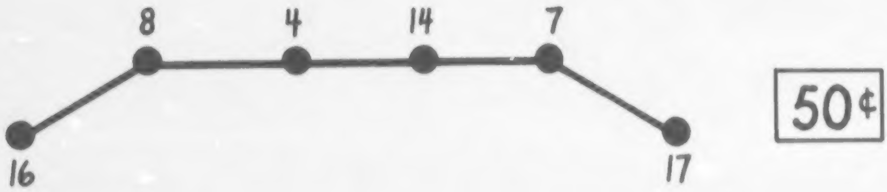
14 to 3



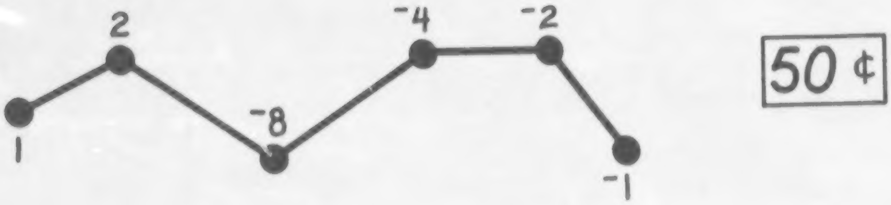
7 to 8



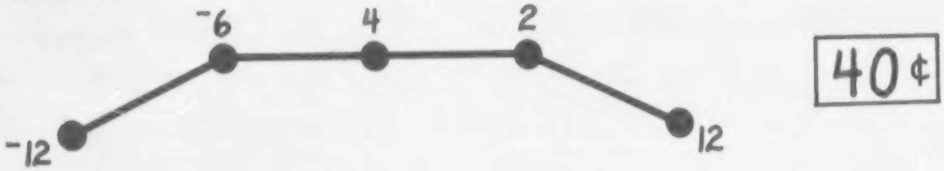
16 to 17



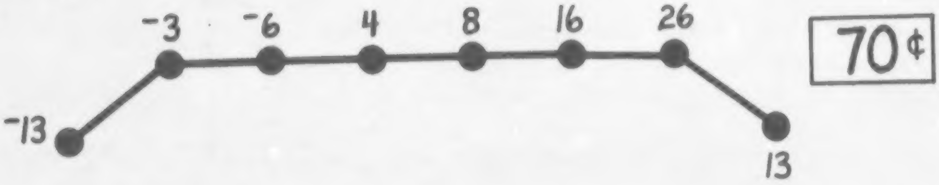
1 to $^-1$



$^-12$ to 12



$^-13$ to 13


$^-13$
$^-3$
$^-6$
4
8
16
26
13
70¢

15 to 16

No Solution. If you look at paths leading from 15, you should notice that the ones' digit of any number that 15 can call is either 0 or 5. On the other hand, the ones' digit of any number that 16 can call is never 0 or 5. Thus 15 cannot talk to 16.

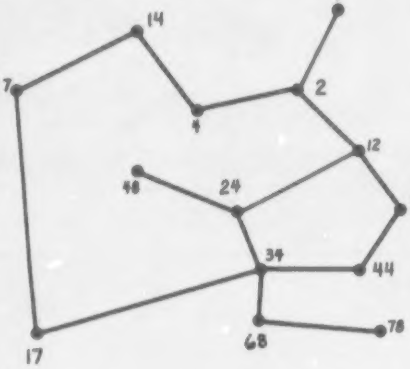
Name ______________________

N 55 *

Telephone Game

First clue

Zip is in this picture. Its dot is not labeled.


1
14
7
2
4
12
48
24
22
34
44
68
78
17

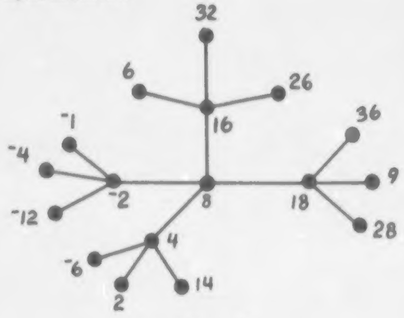
Zip could be 2, 14, 17, 44, 22, 24, 68.

(over)

N 56 *

Second clue

Zip can call 8 for 20¢.

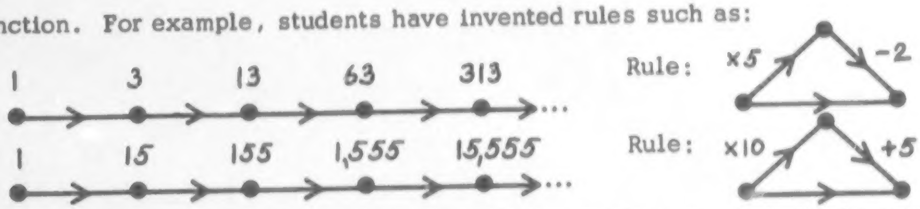

32
6
26
16
36
-1
-4
-2
8
18
9
-12
28
4
-6
14
2

Zip is 14 or 2.

5

GUESS MY RULE

The basic idea of the Guess My Rule game is that the teacher, and later the student, invents new functions (rules) which the class must determine by knowing how they act on certain numbers. The class cooperates in attempting to determine the "secret" rules. The "new" functions are usually, but not necessarily, compositions of the standard arithmetic ones. Often the hints given are in the form of a sequence of numbers generated by the secret function. For example, students have invented rules such as:



In the context of working with these "new" functions, a great deal of mental arithmetic, equation solving and pattern recognition is accomplished in an exciting setting.

CAPSULE LESSON SUMMARIES

Section 5.1: Use arrow pictures to solve linear equations such as

$$(\square \times 2) - 3 = 175$$

and

$$[(\square \times 2) - 10] + 2 = 75$$

Introduce the Guess My Rule game using the function × 2 followed by − 3.

(Worksheets N57, N58, N59)

Section 5.2: Play the Guess My Rule game using the following sequences:

0; 1; 3; 6; 10; 15; 21; ...

(0; 0+1; 1+2; 3+3; 6+4; 10+5; 15+6; ...)

and

1; 4; 9; 16; 25; 36; 49; ... sequence of squares.

Continue with the students inventing the rules.

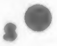
5.1 Guess My Rule #1

Give yourself and each student a hand-calculator.*

Exercise 1

T: I have a secret number on the display of my calculator. I will call it "s".

Draw a dot and label it "s".



T: I will press $\boxed{\times}$ $\boxed{2}$.

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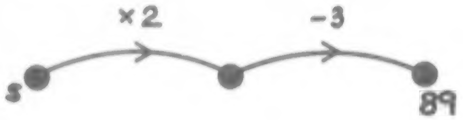
* Because the lesson is concerned with the notion of the return arrow of a given arrow rather than calculations per se, we chose to use hand-calculators to remove the burden of mechanical computation. This frees the student to concentrate on the thinking that is necessary to deal with the situation at hand. Should you not have access to calculators we suggest you omit Exercise 3 and modify Exercises 1 and 2 in the following manner:

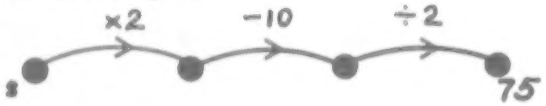
Exercise 1

Follow the same format but use a smaller number to make the computations easier; for example,

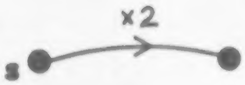


Exercise 2

Follow the same format but use a smaller number to make the computations easier; for example,

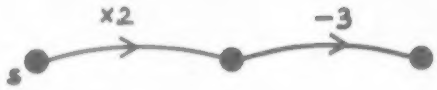


Draw this ×2 arrow.



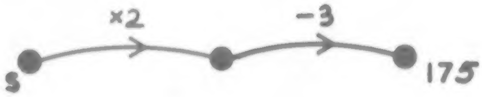
T: Then press $\boxed{-}$ $\boxed{3}$.

Draw this -3 arrow.



T: Then press $\boxed{=}$. Now 175 is on my display.

Label the dot for 175, as below.

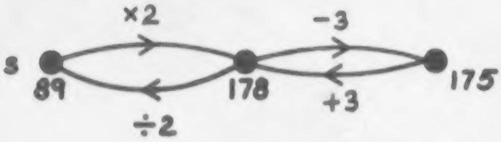


T: How can we find the secret number?

Let a student come to the board and trace the return arrows.

Stress the use of return arrows by drawing the +3 and ÷2 return arrows in different colors than those used so far.

The students should conclude that s is 89.



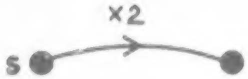
Exercise 2

T: I have put another secret number, s, on the display of my calculator.

Draw a dot and label it "s".

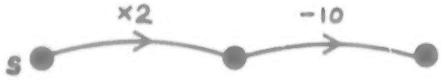
T: I press $\boxed{\times}$ $\boxed{2}$.

Draw a ×2 arrow.



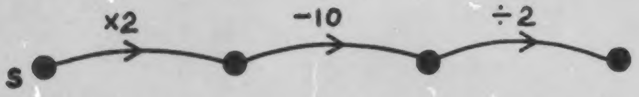
T: Then press $\boxed{-}$ $\boxed{1}$ $\boxed{0}$.

Draw a −10 arrow.



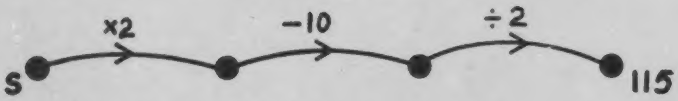
T: Then press $\boxed{\div}$ $\boxed{2}$.

Draw a ÷2 arrow.



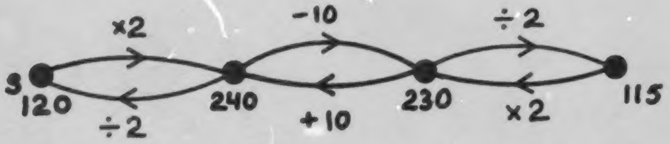
T: Press $\boxed{=}$ and 115 is on my display.

Label the dot for 115.



T: How can we find the secret number?

Let students come to the board, trace return arrows, and label the dots. They should conclude that s is 120.

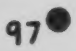

×2
−10
÷2
s
120
240
230
115
÷2
+10
×2

Stress the use of return arrows by drawing them as they are suggested.

Exercise 3

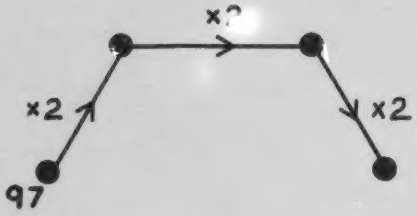
T: Let's put 97 on the display of our calculators.

Draw a dot and label it "97".



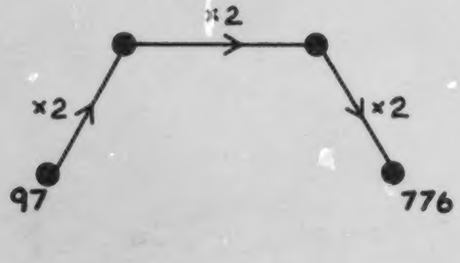
T: Then press $\boxed{\times}\ \boxed{2}$, $\boxed{\times}\ \boxed{2}$, $\boxed{\times}\ \boxed{2}\ \boxed{=}$.[*]

Draw three × 2 arrows as below.



T: What number is on the display? (Answer: 776)

Label the dot for 776.



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[*] See Section 2f of the General Introduction for a discussion on hand-calculator characteristics concerning automatic constants.

T: Now hide the display and try to go back to 97. If you get it, raise your hand. How did you go back to 97?

(Answer: $\boxed{\div}$ $\boxed{2}$, $\boxed{\div}$ $\boxed{2}$, $\boxed{\div}$ $\boxed{2}$ $\boxed{=}$; or $\boxed{\div}$ $\boxed{8}$ $\boxed{=}$)

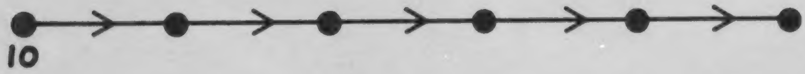
If $\boxed{\div}$ $\boxed{8}$ $\boxed{=}$ is not suggested, ask:

T: Can we go back to 97 by pressing only three buttons?

It is likely that some will suggest $\boxed{\div}$ $\boxed{6}$. Have them try pressing $\boxed{7}$ $\boxed{7}$ $\boxed{6}$ $\boxed{\div}$ $\boxed{6}$ $\boxed{=}$ and see that the display reads "129.33333" and not "97".

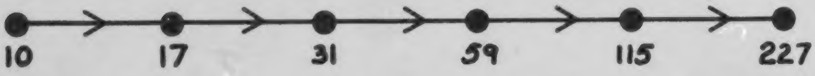
Exercise 4

Draw this arrow picture.



T: I have drawn an arrow picture on the board, but I will not tell you what the red arrows are for. You must try to guess; it's my secret rule.

Slowly and progressively label the dots as in this completed arrow picture.



Pause after you label each dot to allow students to think about your rule and to make some guesses. Suppose +7 is suggested.

T: Does 10 + 7 = 17? (Yes) Does 17 + 7 = 31? (No) So +7 cannot be my secret rule.

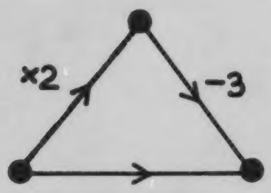
Continue in this manner for each incorrect rule. It may happen that no one guesses the rule. In that case, ask:

T: Could my rule be ×2? Let's see. We start with 10 so 10 × 2 = 20; 17 × 2 = 34; 31 × 2 = 62; and so on.

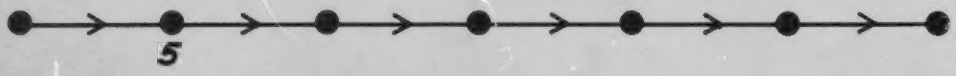
It should be clear that each new number is 3 more than it should be. It is likely that some students will suggest that the secret rule for the red arrow is × 2 followed by − 3.

T: Very good. My rule is × 2 followed by − 3. I will draw another picture on the board as a reminder.

Draw this.



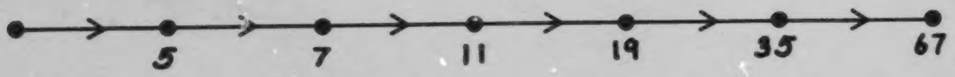
Now draw this arrow picture and ask the students to complete it.



Emphasize the rule by asking students to come to the board, to give the calculations, and to label the dots.

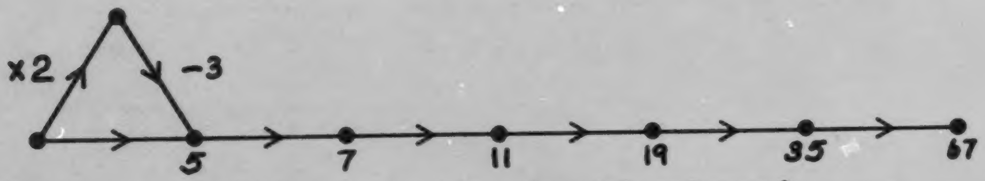
S: 5 × 2 = 10 and 10 − 3 = 7. (Labels the dot to the right of 5, "7".)

The partially completed picture should look like this.



It is probable that the dot on the left will be difficult for many students. If so, proceed in this manner.

Draw these arrows in your picture.



T: How could I find the starting number of the red arrow?

S: Use return arrows.

T: What return arrows are needed?

S: + 3.

S: ÷ 2.

Trace the return arrows as students tell you what they are.

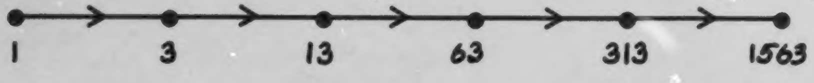
T: Here we have 5 (point to its dot). What is the starting number?

S: 4.

S: 5 + 3 = 8 and 8 ÷ 2 = 4. So it's 4.

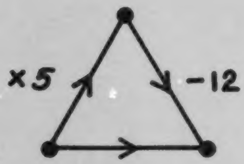
If time permits, let a student invent a new rule for the class to guess. Use an arrow picture to record the results of applying the rule. This is a very open activity with many possibilities. You may be surprised by the degree of sophistication that students will exhibit. Below are a few examples of rules that some of the students in our development classes invented.

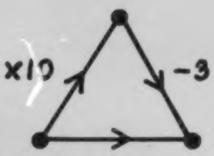
One student drew this arrow picture on the board. The dots were labeled slowly and progressively to give the other students a chance to think of the rule.

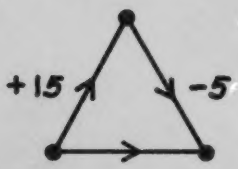


After several guesses, a student suggested the rule: × 5 followed by − 2.

Three other rules were:





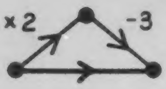


Some students noted that the last rule, + 15 followed by − 5, is the same as + 10.

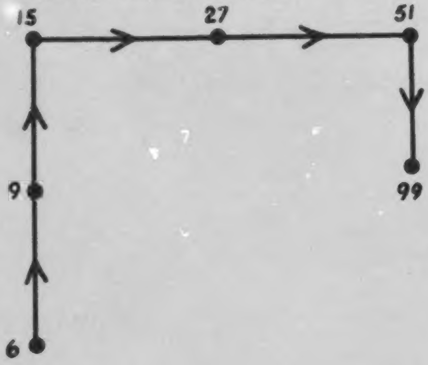
Worksheets N57, N58, and N59 are available. After a while draw the arrow picture from N57 on the board and have students label the dots.

Name ______________________ N 57 *

Here is the code for the red arrows.

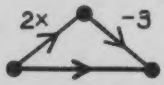


Using this code label the dots in the picture below.

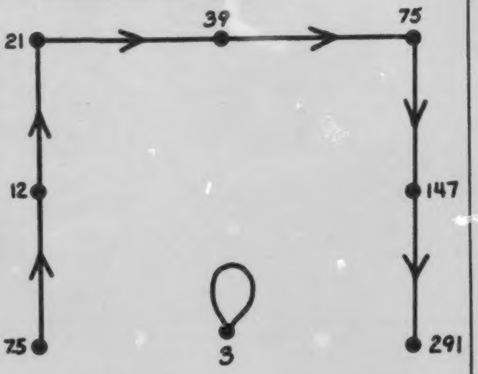

15
27
51
9
99
6

Name ______________________ N 58 **

Here is the code for the red arrows.

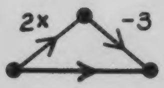


Using this code label the dots in the picture below.

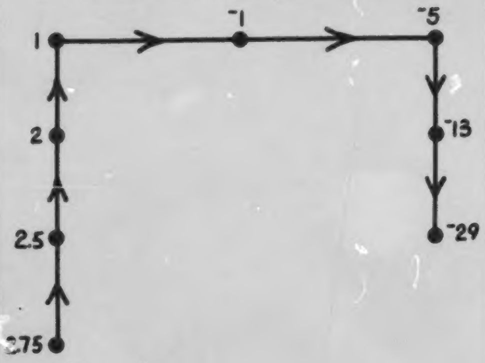

21
39
75
12
147
75
3
291

Name ______________________ N 59 ***

Here is the code for the red arrows.



Using this code label the dots in the picture below.


1
⁻1
⁻5
2
⁻13
2.5
⁻29
2.75

5.2 Guess My Rule #2

Exercise 1

T: Today you will have to guess my secret rule. To help you guess my rule I will write a sequence of numbers on the board.

Slowly write the sequence below on the board. After you have written a few numbers let the class try to predict the next number.

0; 1; 3; 6; 10; 15;

Below, we recount the way this exercise developed in one of our classes.

T: Look carefully at the first six numbers that I have written. What number do you think would come next in my sequence?

S: 20.

T: No. Think carefully.

S: 21.

T: Yes. What next?

S: 28.

Continue writing numbers until it is evident that many of the students have guessed your rule. Then ask:

T: Who can explain my rule?

S: Add 1, then add 2, then add 3, then add 4, and so on.

T: Very good. Now write the next four numbers in my sequence on your paper.

Let the students have a few minutes to work. Then ask a student to write the next four numbers of the sequence on the board. Suppose your list looks like this.

0; 1; 3; 6; 10; 15; 21; 28; 36; 45; 55; ...

T: Here is another way of thinking about my sequence.

Below the first three terms of the sequence write the following calculations.

0; 1; 3; 6; 10; 15; ...
0 × 1; 1 × 1; 1 × 3;

T: What should I write below "6"?

S: 2 × 3.

Add this to your list.

T: What should I write below "10"?

S: 2 × 5.

T: What comes next?

S: 3 × 5.

S: Then 3 × 7.

S: Then 4 × 7.

T: Who can explain my rule?

S: The first part goes 0, 1, 1, 2, 2, 3, 3, 4, 4, and so on and the second part goes 1, 1, 3, 3, 5, 5, 7, 7, and so on.

T: Excellent. What should come next?

S: 6 × 6.

T: 6 × 6 = 36, but it does not fit my pattern.

S: 4 × 9.

T: Good. Write the next two calculations on your paper.

Let a student add these to the list on the board. At this point your sequence should look something like this.

0;	1;	3;	6;	10;	15;	21;	28;	36;	45;	55;	...
0 × 1;	1 × 1;	1 × 3;	2 × 3;	2 × 5;	3 × 5;	3 × 7;	4 × 7;	4 × 9;	5 × 9;	5 × 11;	...

Exercise 2

T: Here is another sequence. Can you guess my rule?

Slowly write the sequence below on the board. After you have written a few numbers, let the class try to predict the next number.

1; 4; 9; 16; 25

T: Look carefully at the first five numbers that I have written. What number do you think would come next in my sequence?

S: 36.

T: Good. What comes after that?

S: 49.

S: Then 64.

S: Then 81.

Continue in this manner until it is evident that many of the students have guessed your rule. Then ask:

T: Who can explain my rule?

S: Add 3; then add 5, then add 7, then add 9, and so on.

T: Good. Write the next three numbers in my sequence on your paper.

After the students have had a few minutes to work, ask a student to write the next three numbers of the sequence on the board. Suppose your list looks like this.

1; 4; 9; 16; 25; 36; 49; 64; 81; 100; 121; 144; ...

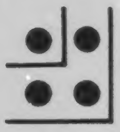
T: Here is another way to think about this pattern. First, I start with 1.

16 N

Draw one dot on the board.

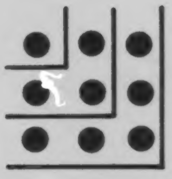
T: Then add 3.

Add three dots to your picture on the board as follows.



T: Then add 5.

Now add five dots as follows.



Continue in this manner as long as you deem necessary.

T: This picture suggests that these numbers could be called square numbers. What do you think of this pattern?

Write these calculations under the terms of the sequence you have on the board.

1; 4; 9; 16; 25; 36; ...

1 × 1; 2 × 2; 3 × 3;

T: What comes next?

S: 4 × 4.

S: Then 5 × 5.

S: Then 6 × 6. And so on.

Exercise 3

Write the first five numbers of the sequence below on the board.

1; 3; 7; 15; 31

T: What number should come next in my sequence?

S: 63.

S: Then 127.

Continue writing numbers until it is evident that several students have guessed your rule. Then ask:

T: Who can explain my rule?

S: Add 2, then add 4, then add 8, then add 16, and so on.

S: Or it could be 2 × followed by + 1.

T: Very good. My rule could be either of those.

T: I would like someone to invent a rule and let the class try to guess it.

Let the students have a few minutes to think of their rules. Encourage them to write several numbers of their sequences on paper. Walk around and monitor student progress. You will probably be surprised by the variety and sophistication of the rules. Below, we describe two rules that were offered in our development classes.

A student wrote this sequence on the board.

5; 8; 14; 26; 50;

S: Your rule is 2 × followed by -2.

S: That works, but I was adding 3, then adding 6, then adding 12, then adding 24, and so on.

Another student wrote this very interesting sequence on the board.

1; 15; 155; 1,555; 15,555;

S: Just add a "5" to the end.

S: Yes, but I was multiplying by 10 then adding 5.

Continue with this activity as long as there is student interest.

6

SUBTRACTION

The lessons of this chapter provide a variety of problems that reinforce and deepen the subtraction experiences of the CSMP Upper Primary Grades Program. Negative numbers are used in both addition and subtraction problems, and the experiences of relating subtraction with addition are extended. Properites of subtraction are used to generate patterns and these patterns suggest shortcuts in calculation. For example,

425 - 103 = 322		39 - 20 = 19
425 - 100 = 325	and	40 - 21 = 19
425 - 97 = 328		44 - 25 = 19
. . .		. . .

The subtraction algorithm is reviewed and reinforced by these new experiences.

CAPSULE LESSON SUMMARIES

Section 6.1:	Label the arrows of an extensive arrow road with addition and subtraction functions where there are dots for both positive and negative integers. Examine subtraction properties in patterns generated by related subtraction statements. (Worksheets N60, N61, N62)
Section 6.2:	Continue to develop situations similar to those of Section 6.1. Review the subtraction algorithm and use it to solve several subtraction problems. (Worksheets N63, N64, N65)

6.1 Subtraction #1

Exercise 1

T: How much smaller is 4 than 10?

S: 6.

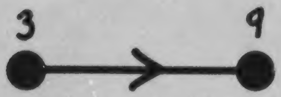
Continue in this manner for (Answers are in the boxes.)

4 than 20?	16
4 than 50?	46
4 than 100?	96
4 than 1,000?	996
4 than 2,000?	1,996
4 than 10,000?	9,996
23 than 30?	7
23 than 50?	27
23 than 100?	77
23 than 200?	177
23 than 1,000?	977

Exercise 2

T: I shall draw an arrow road on the board. You are to tell me what each blue arrow could be for.

Draw this picture on the board.

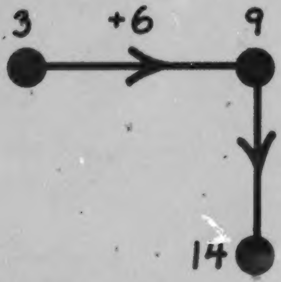


T: What could the blue arrow be for?

S: + 6.

S: × 3.

Label the arrow and extend the picture as shown below.

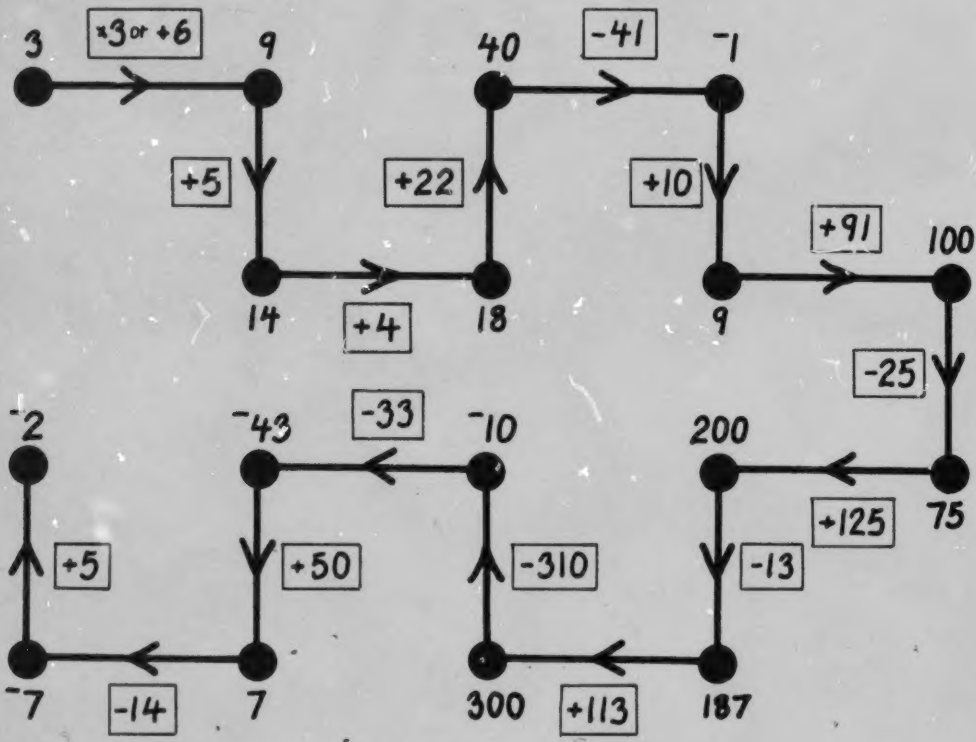


It may be necessary, at this point, to remind students that here we will use a blue arrow for many different operations.

T: What could the blue arrow be for?

S: + 5.

Continue in this way as long as there is student interest. Feel free to adjust the level of difficulty to suit your class. The following picture shows an arrow road that was used in one of our development classes. Student responses are in the boxes.


3
×3 or +6
9
40
-41
⁻1
+5
+22
+10
+91
100
14
+4
18
9
-25
⁻2
⁻43
-33
⁻10
200
75
+125
+5
+50
-310
-13
⁻7
-14
7
300
+113
187

Exercise 3

Write this on the board.

33 - 9 33 - 10

T: Which is larger?

S: 33 - 9 is larger.

T: Why?

S: You are subtracting a smaller number.

T: How much larger is 33 - 9 than 33 - 10?

S: One larger.

T: What is the second number?

S: 23.

T: So what must the first number be?

S: 24, because it is one more.

Continue with this exercise for the following pairs of numbers. The answers are in the boxes and should not be written on the board.

$$45 - 18 = \boxed{27} \qquad 38 - 39 = \boxed{^{-}1}$$

$$45 - 20 = \boxed{25} \qquad 39 - 38 = \boxed{1}$$

$$275 - 100 = \boxed{175} \qquad 2{,}472 - 1{,}000 = \boxed{1{,}472}$$

$$275 - 97 = \boxed{178} \qquad 2{,}472 - 994 = \boxed{1{,}478}$$

Exercise 4

Write this set of problems on the board.

$$76 - 10 =$$

$$76 - 11 =$$

$$76 - 9 =$$

T: Solve these problems. Try to do as little work as possible.

If there are some students who have not as yet caught on to the pattern, try asking:

T: Who can tell me the answer to the first problem?

S: 66.

T: Good. Is the answer to the second problem larger or smaller than 66?

S: Smaller.

T: How much smaller?

S: One smaller.

T: So what number is 76 - 11?

S: 65.

T: What about the third problem? Is it larger or smaller than 66?

S: It is one larger.

S: 76 - 9 = 67.

Continue this activity with the following problems. (Answers are in boxes and should not be written on the board.)

$97 - 20 = \boxed{77}$

$97 - 22 = \boxed{75}$

$97 - 18 = \boxed{79}$

$425 - 100 = \boxed{325}$

$425 - 103 = \boxed{322}$

$425 - 97 = \boxed{328}$

$5{,}675 - 1{,}000 = \boxed{4{,}675}$

$5{,}675 - 1{,}001 = \boxed{4{,}674}$

$5{,}675 - 999 = \boxed{4{,}676}$

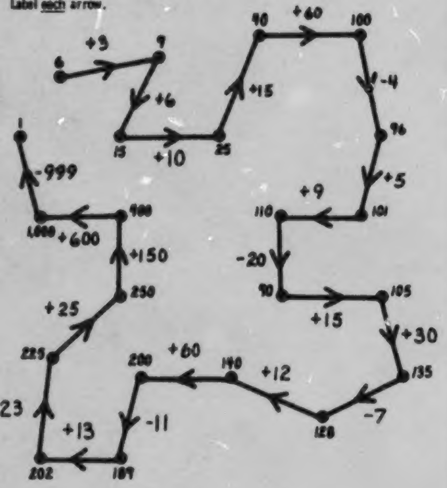
Worksheets N60, N61, and N62 are available for the rest of the lesson.

It may be necessary to remind the students that in these problems the arrows all having the same color does not mean that they are all for the same thing.

Name

N 60 *

Label each arrow.

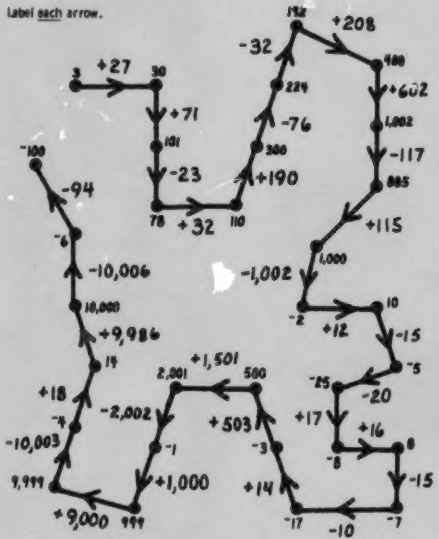

6
+3
9
+6
15
+10
25
+15
90
+60
100
-4
96
+5
101
+9
110
-20
90
+15
105
+30
135
-7
128
+12
140
+60
200
-11
189
+13
202
+23
225
+25
250
+150
400
+600
1,000
-999
1

Other solutions are possible.

Name

N 61 **

Label each arrow.


3
+27
30
+71
101
-23
78
+32
110
+190
300
-76
224
-32
192
+208
400
+602
1,002
-117
885
+115
1,000
-1,002
-2
+12
10
-15
-5
-20
-25
+17
-8
+16
8
-15
-7
-10
-17
+14
-3
+503
500
+1,501
2,001
-2,002
-1
+1,000
999
+9,000
9,999
-10,003
-4
+18
14
+9,986
10,000
-10,006
-6
-94
-100

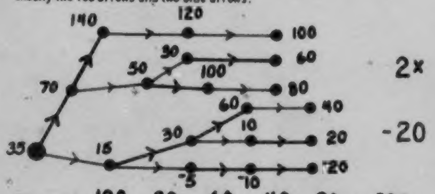
Other solutions are possible.

Name

N 62 ***

First clue

Pick is the ending number of a red-blue road starting at 35 and using exactly two red arrows and two blue arrows.


140
120
100
30
60
50
100
70
80
2×
60
40
10
30
20
-20
35
15
-20
-5
-10

Pick could be 100 ; 80 ; 60 ; 40 ; 20 or -20 .

Second clue

Pick is not a positive divisor of 1,000.

Pick could be 80 ; 60 or -20 .

Third clue

Pick is smaller than 50.

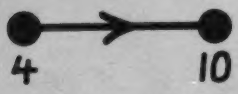
Pick is -20 .

6.2 Subtraction #2

Exercise 1

T: I shall draw an arrow road on the board. You are to tell me what each red arrow could be for.

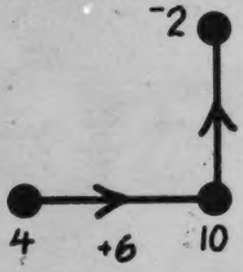
Start the road with this arrow.



T: What could this red arrow be for?

S: + 6.

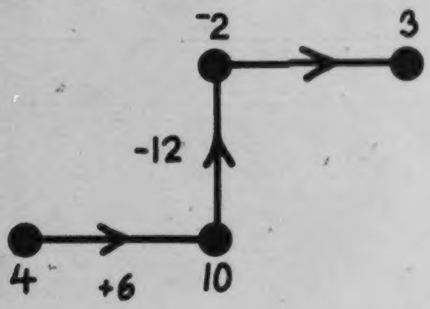
Label the arrow "+ 6" and add this arrow to your drawing.



T: What could this red arrow be for?

S: - 12.

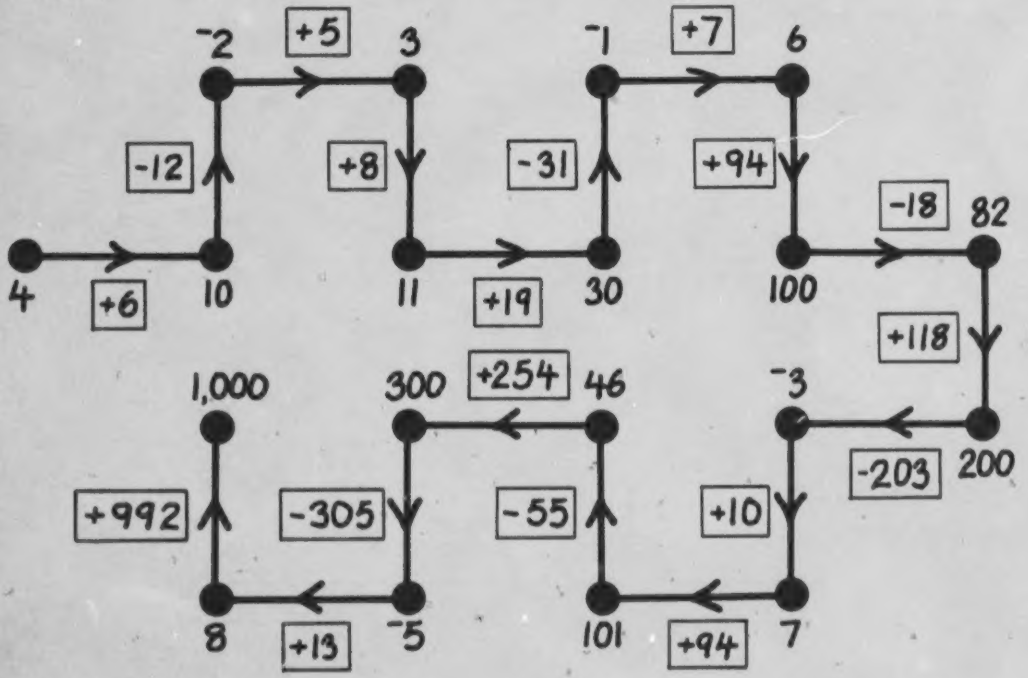
Again, label the arrow and add this arrow to the road.



T: What could this arrow be for?

S: + 5.

Continue labeling and extending as long as there is student interest. Feel free to adjust the level of difficulty to suit your class. The arrow road below was drawn in one of our development classes. Student responses are in the boxes.


$^{-}2$
+5
3
$^{-}1$
+7
6
-12
+8
-31
+94
-18
82
4
+6
10
11
+19
30
100
+118
1,000
300
+254
46
$^{-}3$
-203
200
+992
-305
-55
+10
8
+13
$^{-}5$
101
+94
7

Exercise 2

Write these problems on the board.

39 - 20
40 - 21
44 - 25
46 - 27

T: Why did I write these problems next to each other on the board?

A likely answer is that each is a name for 19.

T: Do you see a pattern to these problems?

Accept any comment that is reasonable. If the following pattern is not suggested, suggest it yourself.

T: Look closely at "40 - 21" and "44 - 25".
(Point to them on the board.)
40 + 4 = 44 and 21 + 4 = 25.
What could we do to "40 - 21" to obtain "46 - 27"?

S: Add 6 to 40 and add 6 to 21.

T: What could be done to "39 - 20" to obtain "44 - 25"?

S: Add 5 to 39 and add 5 to 20.

T: What could be done to "46 - 27" to obtain "39 - 20"?

S: Subtract 7 from 46 and subtract 7 from 27.

This activity may be extended if necessary.

T: In the problems I am going to write on the board, the number is always 19. You find what number is missing in the box. Use our pattern to help you.

Have the students work these problems on their own papers.

$\square - 30 = 19$
$59 - \square = 19$
$70 - \square = 19$

$\square - 60 = 19$
$\square - 100 = 19$
$200 - \square = 19$

$\square - 200 = 19$
$1{,}000 - \square = 19$

After the class has had time for individual work on these problems, check them orally or have students write their answers on the board. For some of the problems ask students how they arrived at their answers. For example,

S: I used "39 - 20 = 19" to get "$\boxed{49}$ - 30 = 19"; I added 10 to 20 and added 10 to 39 to obtain 49.

Be sure to accept any reasonable explanation, even if it is not in the spirit of the above. Solutions to the problems are given below.

$\boxed{49} - 30 = 19$

$59 - \boxed{40} = 19$

$70 - \boxed{51} = 19$

$\boxed{79} - 60 = 19$

$\boxed{119} - 100 = 19$

$200 - \boxed{181} = 19$

$\boxed{219} - 200 = 19$

$1{,}000 - \boxed{981} = 19$

Exercise 3

Write this on the board.

$$\overset{8}{236}$$

T: What is this number?

If no one volunteers an answer, suggest:

T: We are adding 8 ones to 236.

Do the calculation on the board.

$$\overset{8}{236} = 236 + 8 = 244$$

Next, write this on the board.

$$\overset{7}{245}$$

T: What number is this?

S: 245 plus 7 tens.

S: 245 + 70 = 315.

T: Copy these problems on your paper and solve them.

Write this set of problems on the board.

$24\overset{10}{5} = \boxed{\quad\quad}$ $\quad$ $2\overset{10}{4}9 = \boxed{\quad\quad}$ $\quad$ $2,\overset{6}{4}5\overset{1}{9} = \boxed{\quad\quad}$

$2\overset{7}{3}7 = \boxed{\quad\quad}$ $\quad$ $\overset{7}{3}65 = \boxed{\quad\quad}$

Check these problems collectively. The answers are given below.

$24\overset{10}{5} = \boxed{255}$ $\quad$ $2\overset{10}{4}9 = \boxed{349}$ $\quad$ $2,\overset{6}{4}5\overset{1}{9} = \boxed{3,060}$

$2\overset{7}{3}7 = \boxed{307}$ $\quad$ $\overset{7}{3}65 = \boxed{1,065}$

Write this on the board.

$62\overset{10}{4}$ $\quad\quad$ $6\overset{1}{2}4$

T: Compare these two numbers. Which is larger?

S: They are equal.

S: In the first you add 10 ones and in the second 1 ten.

S: They both equal 634.

Try a similar exercise with

$$2{,}5\overset{10}{3}7 \qquad 2{,}\overset{1}{6}37$$

In this case,

$$2{,}5\overset{10}{3}7 < 2{,}\overset{1}{6}37$$

because

$$2{,}5\overset{10}{3}7 = 2{,}637 \quad \text{and} \quad 2{,}\overset{1}{6}37 = 2{,}737$$

Exercise 4

Write this problem on the board.

$$\begin{array}{r} 872 \\ \underline{-545} \end{array}$$

T: My friend, Nabu, has a way of working subtraction problems which is probably not like your way.* To subtract, he first adds the same number to each number like this.

Write the red numerals on the board.

$$\begin{array}{r} 87\overset{10}{2} \\ \underline{-5\overset{1}{4}5} \end{array}$$

---

*This is not true for a student that has been in the CSMP curriculum since the third grade, since this is the only subtraction algorithm used in the CSMP Upper Primary Grades Program.

T: Since subtracting 5 from 2 would give a negative number, Nabu first adds 10 ones to 872 and 1 ten to 545. Then he subtracts 5 from 12, 5 from 7, and 5 from 8.

$$\begin{array}{r} 87\overset{10}{2} \\ -5\overset{1}{4}5 \\ \hline 327 \end{array}$$

Write this problem on the board and let a student explain how to solve it.

$$\begin{array}{r} 429 \\ -136 \\ \hline \end{array}$$

Solution:

$$\begin{array}{r} 4\overset{10}{2}9 \\ -\overset{1}{1}36 \\ \hline 293 \end{array}$$

T: Now, all of you solve these problems on your paper.

$$\begin{array}{r} 691 \\ -345 \\ \hline \end{array} \qquad \begin{array}{r} 580 \\ -348 \\ \hline \end{array} \qquad \begin{array}{r} 2{,}749 \\ -1{,}287 \\ \hline \end{array}$$

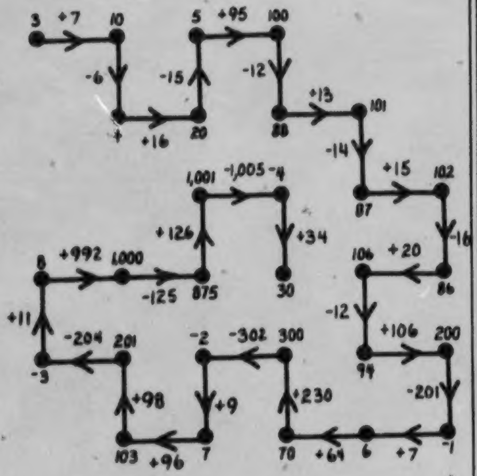
Have three students work these problems at the board while the rest of the class checks their work.

$$\begin{array}{rcrcr} 69\overset{10}{1} & \qquad & 58\overset{10}{0} & \qquad & 2{,}7\overset{10}{4}9 \\ -3\overset{1}{4}5 & & -3\overset{1}{4}8 & & -1{,}\overset{1}{2}87 \\ \hline 346 & & 232 & & 1{,}462 \end{array}$$

Worksheets N63, N64, and N65 are available for the rest of the lesson.

Name ______________________ **N 63** *

Label each arrow.


3
+7
10
-6
4
+16
20
-15
5
+95
100
-12
88
+13
101
-14
87
+15
102
-16
86
+20
106
-12
94
+106
200
-201
-1
+7
6
+64
70
+230
300
-302
-2
+9
7
+96
103
+98
201
-204
-3
+11
8
+992
1,000
-125
875
+126
1,001
-1,005
-4
+34
30

Other solutions are possible.

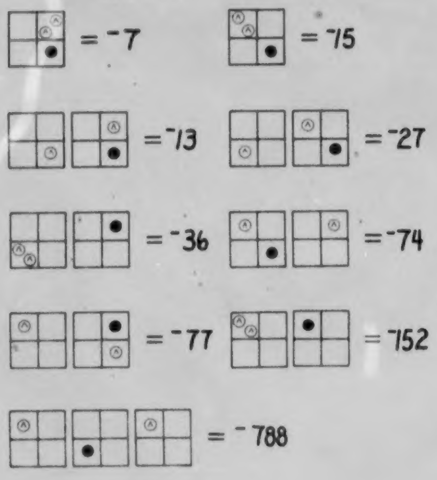
Name ______________________ **N 64** **

Each of these is a different name for the same number. Complete them.

51 − 28	71 − $\boxed{48}$	101 − $\boxed{78}$
31 − $\boxed{8}$	$\boxed{23}$ − 0	100 − $\boxed{77}$
81 − $\boxed{58}$	$\boxed{123}$ − 100	1,000 − $\boxed{977}$
201 − $\boxed{178}$	$\boxed{1{,}023}$ − 1,000	20,000 − $\boxed{19{,}977}$

Name ______________________ **N 65** ***

Put these numbers on the Minicomputer using exactly one positive checker and two negative checkers.

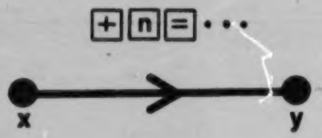

= ⁻7
= ⁻15
= ⁻13
= ⁻27
= ⁻36
= ⁻74
= ⁻77
= ⁻152
= ⁻788

Other solutions are possible.

7

MULTIPLES AND DIVISORS

In this chapter various situations are presented in which the relation "is a positive divisor of" is reviewed and explored more deeply. These experiences reinforce and extend those of Part I of the CSMP Intermediate Grades Program. Properties of and patterns generated by the "is a multiple of" and "is a positive divisor of" relations are investigated with an emphasis on the converse relationship between these relations. The hand-calculator* is used to introduce some new relations such as $\boxed{+}\ \boxed{5}\ \boxed{=}$ ... and its converse $\boxed{-}\ \boxed{5}\ \boxed{=}$ ... . You will discover that



means that y is x plus some positive multiple of n. Many interesting patterns involving these relations will be explored in this chapter.

CAPSULE LESSON SUMMARIES

Section 7.1:	Investigate the relation "is a positive divisor of" in several different arrow pictures situations. Use these situations to find numbers with more than seven positive divisors and to find common multiples of some numbers. (Worksheets N66, N67, N68)
Section 7.2:	Use the hand-calculator to introduce some new relations such as $\boxed{+}\ \boxed{3}\ \boxed{=}$ ... and $\boxed{+}\ \boxed{5}\ \boxed{=}$ ..., and their converses $\boxed{-}\ \boxed{3}\ \boxed{=}$ ... and $\boxed{-}\ \boxed{5}\ \boxed{=}$ ..., respectively. Begin to investigate some patterns in sequences of numbers and in arrow pictures involving these new relations. (Worksheets N69, N70, N71)
Section 7.3:	Continue the experiences of Section 7.2 in order to gain more insight into the hand-calculator relations. Investigate a more complex situation where two of these relations are involved. (Worksheets N72, N73, N74)
Section 7.4:	Continue to investigate situations involving the hand-calculator relations especially to explore how $\boxed{+}\ \boxed{5}\ \boxed{=}$ ... can be used to partition the set of integers. Present another situation in the context of a detective story involving two of these relations. (Worksheets N75, N76, N77, N78)

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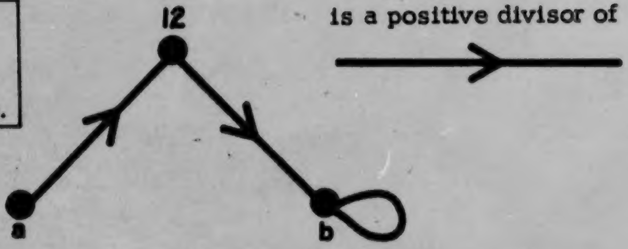
* You may want to reread Section 2(f) of the General Introduction on the Role and Use of Hand-Calculators before teaching the lessons titled "Hand-Calculator Relations".

7.1 The Relation, "Is a Positive Divisor Of"

Exercise 1

Draw this picture on the board.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



T: The red arrow is for "is a positive divisor of". What number could be here (point to a)?

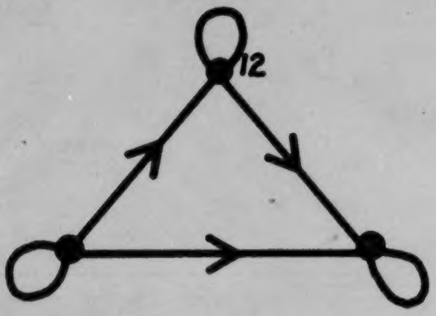
S: 1; 2; 3; 4; or 6 (any positive divisor of 12 except 12).

T: What number could be here (point to b)?

S: 24; 36; 48; 60; . . . (any positive[*] multiple of 12 except 12).

T: There are some red arrows and loops missing from our picture. Who can add these to our picture?

Have a student come to the board and draw the missing arrows and loops. The completed picture should look like this.



---

[*] The loop at b means that b is a positive divisor of b and hence a positive number.

Trace some return arrows in your picture.

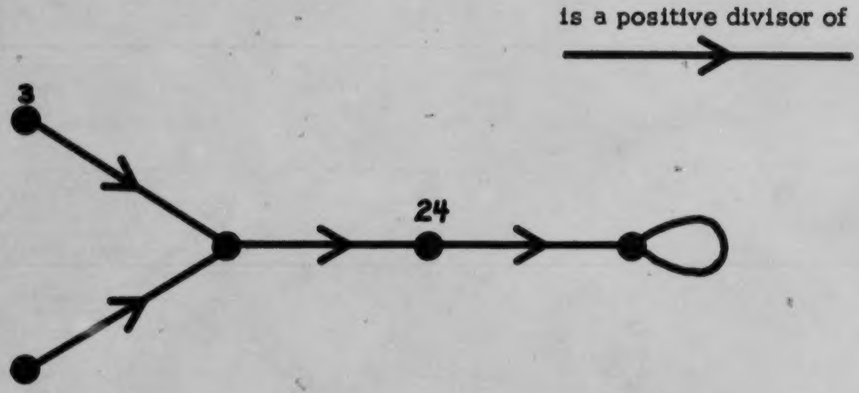
T: What could the return arrows be for?

S: "Is a multiple of".*

Do a similar exercise replacing "12" by "15".

Exercise 2

Draw this arrow picture on the board.

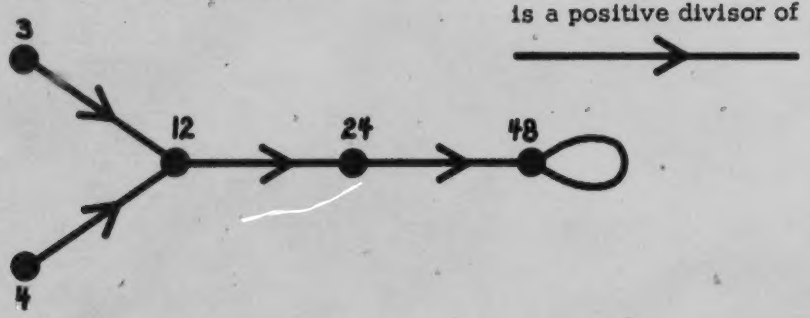


T: Copy this arrow picture, label the dots, and draw the missing arrows and loops. If you finish, think of another solution and draw a new picture.

Give the class several minutes to investigate this situation. Walk around and monitor student progress. After a few minutes have a student come to the board and label the dots. There are many solutions to the problem. Suppose a student labeled the dots in this manner.

---

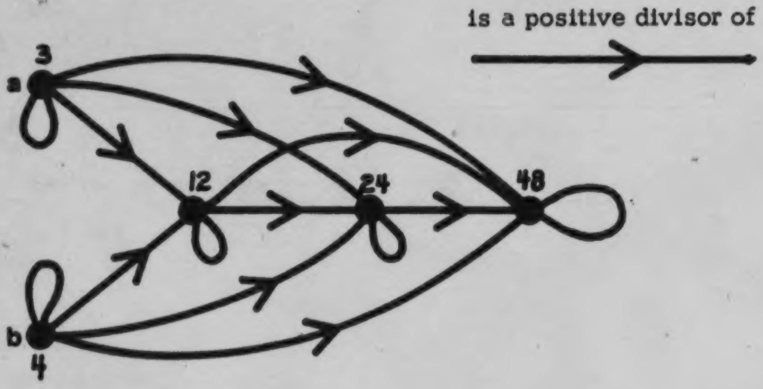
*"Is a positive multiple of" would be another correct answer because the "is a positive divisor of" relation forces all the numbers in this picture to be positive. However, you need not insist that the students observe this subtlety.



T: Very good. Who can draw the missing arrows and loops?

Have students come to the board and draw the arrows. For our example the picture should look like this.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



T: Does everyone's arrow picture look like this one?

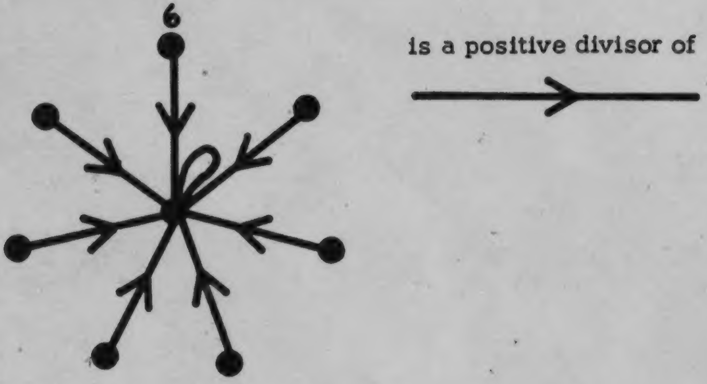
S: No, mine has an arrow from top to bottom (a to b). I had 6 at the bottom (b).

S: I had 3 at the bottom (b) and 6 at the top (a), so my arrow went from bottom to top (b to a).

T: Good. So there are many other possibilities depending on what you labeled the dots.

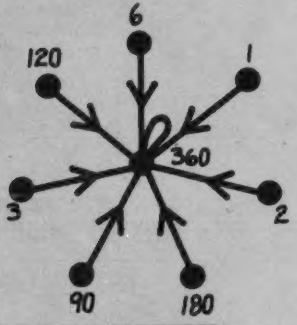
Exercise 3

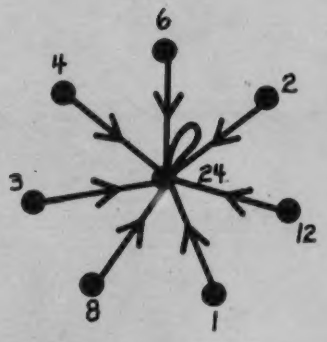
Draw this arrow picture.

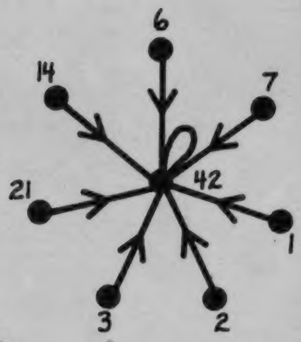


T: Draw this picture and label the dots. All the dots are for different numbers. I have omitted the loops, but we know there would be a loop at every dot.

Walk around and monitor student progress. Some students may label the central dot "12". If this happens, have them count the positive divisors of 12 (there are six: 1, 2, 3, 4, 6, 12) and remind them that the number they are looking for has at least seven positive divisors besides itself. After a few minutes have a student put a solution on the board. Three possible solutions are given below.*







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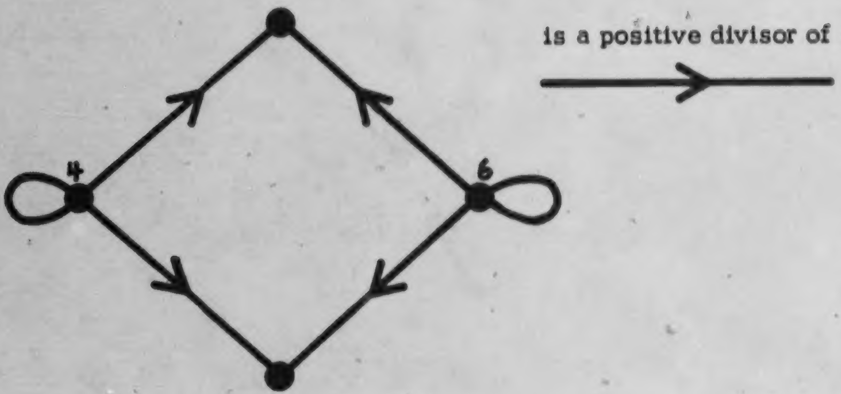
*Not all of the positive divisors of 360 are in the picture, because 360 has more than eight positive divisors.

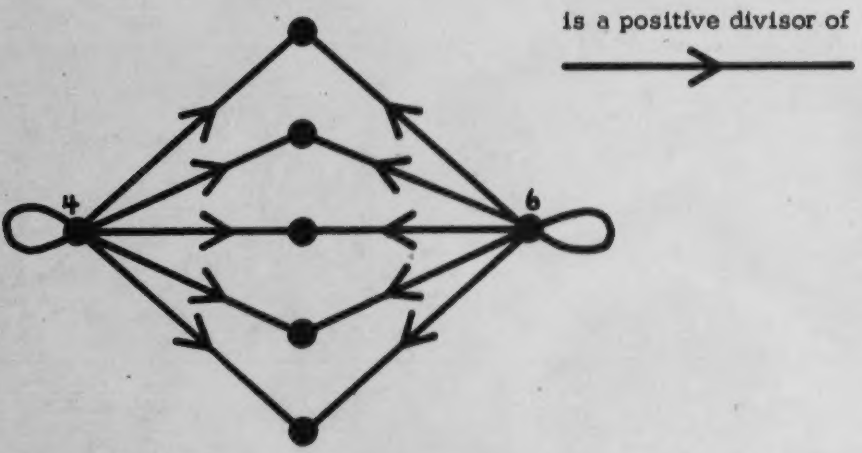
Exercise 4

Draw this arrow picture on the board.

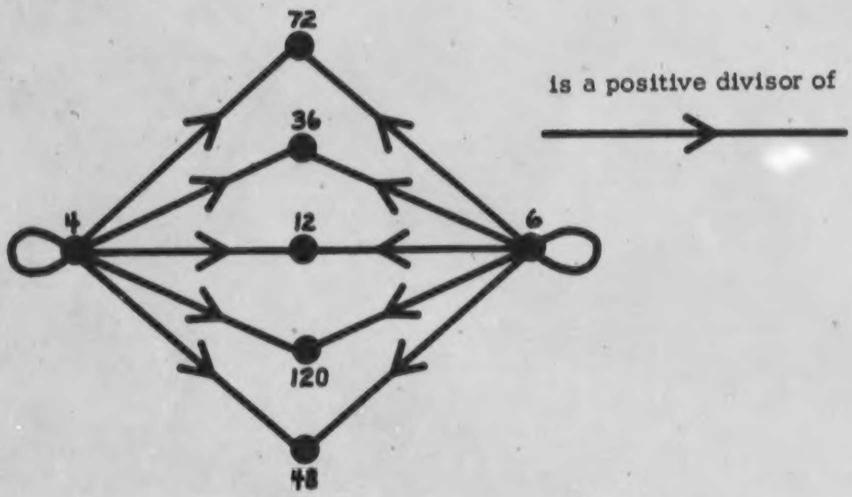

4
6
is a positive divisor of

T: Copy the picture and label the dots.

Give the students a few minutes to work on their own. As the faster students finish, have them add these dots and arrows to their pictures.


4
6
is a positive divisor of

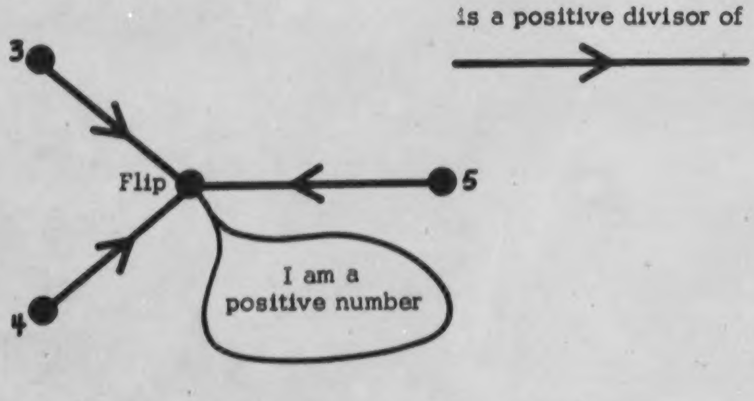
A possible solution:


is a positive divisor of
72
36
12
4
6
120
48

Many other solutions are also possible. Accept any positive multiple of 12 (e.g., 12; 24; 36; 48; ...) or any negative multiple of 12 (e.g., $^{-}12$; $^{-}24$; $^{-}36$; ...) or 0. If no one suggests 0 or a negative multiple of 12, do not insist upon these solutions.

Exercise 5

Draw this arrow picture on the board.


is a positive divisor of
3
Flip
5
4
I am a positive number

T: What are some numbers that Flip could be? Try to find several.

S: 120.

T: Can you think of a smaller number that Flip could be?

S: 60.

T: One smaller than that?

S: 60 is the smallest.

The class should conclude that Flip could be 60, 120, 180, 240, ...; i.e., any positive multiple of 60.

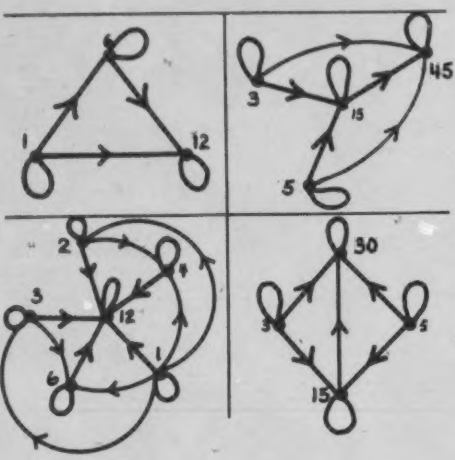
Worksheets N66, N67, and N68 are available for the remainder of the lesson.

Name

N 66 *

Label the dots and draw the missing arrows

is a positive divisor of →

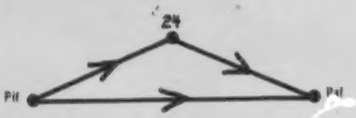

1
12
3
45
15
5
2
4
3
12
6
1
30
3
5
15

Other solutions are possible.

Name

N 67 **

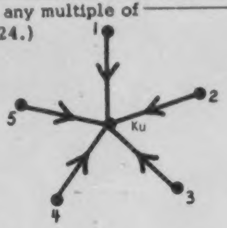
is a positive divisor of →



Pif could be 1 ; 2 ; 3 ; 4 ; 6 ; 8 ; 12 .

Paf could be 48 ; 72 ; 96 ; 120 ; 144 ; 168 ; and so on.

(Paf can be any multiple of 24 except 24.)



Ku could be 60 ; 120 ; 180 ; 240 ; 300 ; 360 ; and so on.

(Ku can be any multiple of 60.)

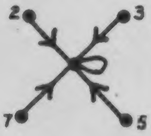
Name

N 68 ***

Zip is a positive integer.

First clue

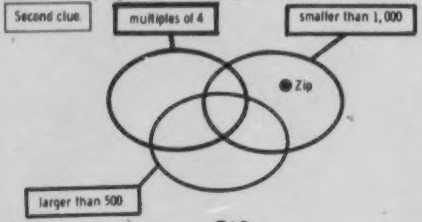
is a positive divisor of →



Zip could be 210 ; 420 ; 630 ; 840 ; 1,050 ; and so on.

(Zip can be any positive multiple of 210.)

Second clue


multiples of 4
smaller than 1,000
Zip
larger than 500

Zip is 210.

7.2 Hand-Calculator Relations #1

Distribute as many hand-calculators as you have available. If there are only a few calculators, have the students with calculators answer any of the questions related to the calculator when the display is exposed. Any of the children can answer when the display is hidden. All students should be encouraged to think carefully and stay just as involved as if they had a calculator with a hidden display. The students with the calculators can check all the answers. For different parts of the lesson, choose different students to work the calculators in order to give as many of them the experience as possible.

Exercise 1

When giving instructions, always do so slowly and clearly.

T: Turn on the calculators. We will start from 13. Put 13 on the display of your calculator. Slowly press $\boxed{+}\ \boxed{3}\ \boxed{=}\ \boxed{=}\ \boxed{=}$ and so on. Read the numbers aloud that appear on the display.

S: 16, 19, 22, 25, 28, 31, 34, ...

T: What is happening?

S: It's counting by 3.

Let the students continue reading the numbers which appear on the display, perhaps until 100 appears.

T: Now clear the display and listen carefully to my instructions. Start with 20 on the display. Press $\boxed{+}\ \boxed{3}\ \boxed{=}\ \boxed{=}\ \boxed{=}$. What number is on the display?

S: 29.

T: Now press $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$. What number is on the display now?

S: 41.

T: Now hide the display until I ask for a calculator check.
Press $\boxed{=}$ $\boxed{=}$ $\boxed{=}$. What number should be on the display?

S: 50.

T: Calculator check.

S: That's right.

T: Hide the display. Press $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$. What number should be on the display?

S: 62.

T: Check.

S: Right.

T: Hide the display and press $\boxed{=}$ $\boxed{=}$. What two numbers should have appeared?

S: 65 and 68.

Let several students whisper in your ear the number of times they think $\boxed{=}$ should be pressed to get from 68 to 80. Ask one student to respond orally. (Four times.)

T: What is the smallest number larger than 100 that will appear?

S: 101.

T: Let's have a calculator check.

Have a "checker" press $\boxed{=}$ and see that 101 does indeed appear.

T: How do you think we could go back in such a way that we would see each number reappear in reverse order?

S: Press $\boxed{-}$ $\boxed{3}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ and so on.

T: All right, we'll do that.

Have the students with calculators read the numbers which appear on the display (101, 98, 95, 92, 89, ...).

T: Without using the calculator pretend we are still pressing $\boxed{=}$. What are some numbers that would appear?

S: 65.

S: 50.

S: 20, the starting number.

T: Can you name some positive numbers smaller than 20 that would appear?

S: 17.

S: 11.

For any answers that are disputed, have a calculator check.

T: What is the smallest positive number that will appear?

S: 2.

T: If we keep pressing $\boxed{=}$ what negative numbers will appear?

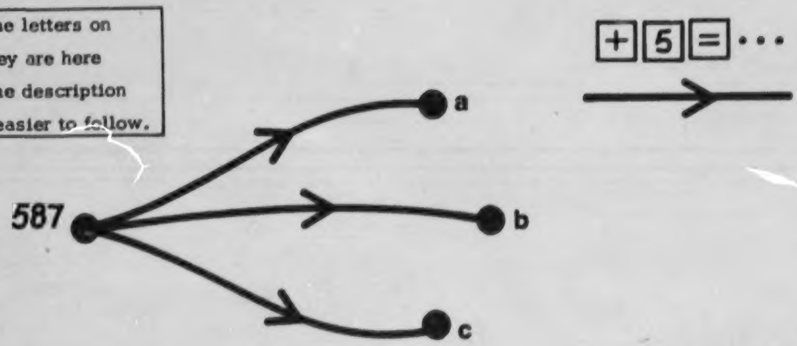
S: ⁻1.

S: ⁻4, ⁻7, ⁻10 and so on.

Exercise 2

Draw this arrow picture on the board.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



T: The red arrow is for $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ $\boxed{=}$ ..., as many times as you want. (Point to "587" and trace the arrow from 587 to a.) We start with 587 on the display and press $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ $\boxed{=}$ and so on. What number could go here (point to a)?

S: 602.

Label a "602" or any other correct response.

T: Press $\boxed{=}$ some more and watch the numbers that appear on the display. What could this dot be for (point to b)?

Let several students give answers. Then label b with one of the correct responses.

T: I want to label this dot (point to c) with a number larger than 1,000. What could it be?

Accept any correct answer and label c.

T: Do you notice any pattern to the numbers that appear on the display?

S: The last digit is always 2 or 7.

T: Can someone tell me a close neighbor of 1,000 that could be in our arrow picture?

S: 1,002 (or 997).

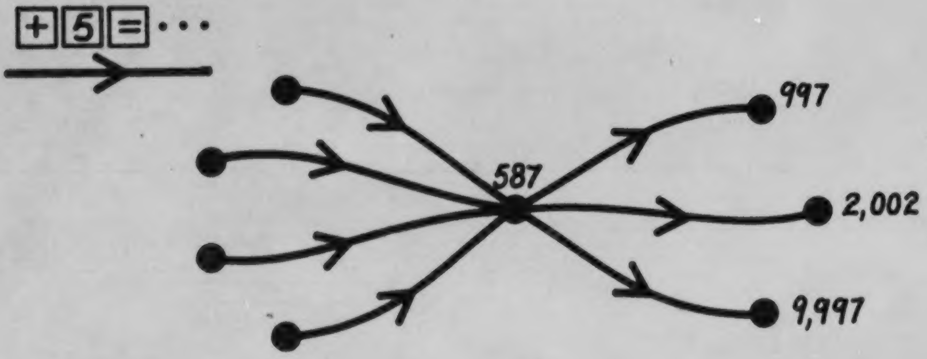
Erase the numerals that are at a, b, and c. Repeat the exercise (still starting with 587) and ask the following questions for a, b, and c.

T (pointing to a): What is the largest number possible that is smaller than 1,000? (Answer: 997)

T (pointing to b): What is the smallest number possible that is larger than 2,000? (Answer: 2,002)

T (pointing to c): What is the largest number possible that is smaller than 10,000? (Answer: 9,997)

Add these arrows to your drawing.


+5=...
997
587
2,002
9,997

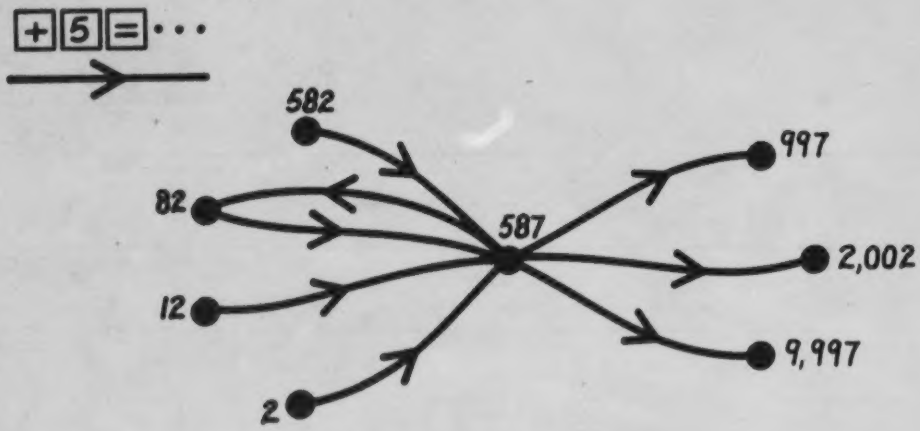
T: What could these dots be for?

Let the students voice their opinions. As correct answers are given, record them on your arrow picture.

T: What is the smallest positive number that we could put in our picture?

S: 2.

Add this return arrow to your picture.


+5=...
582
82
12
2
587
997
2,002
9,997

T: What could this blue arrow be for?

S: $\boxed{-}\ \boxed{5}\ \boxed{=}\ \boxed{=}$ .... .

T: Who can find some negative numbers that could be in our picture?

Let the class have a minute or two to consider the problem. If some students have trouble finding any negative numbers, suggest:

T: Put 2 on the display of your calculator.

Then press $\boxed{-}\ \boxed{5}\ \boxed{=}\ \boxed{=}$ and so on. What numbers appear?

S: $^{-}3$ (or $^{-}8$; $^{-}13$; $^{-}18$; $^{-}23$; etc.).

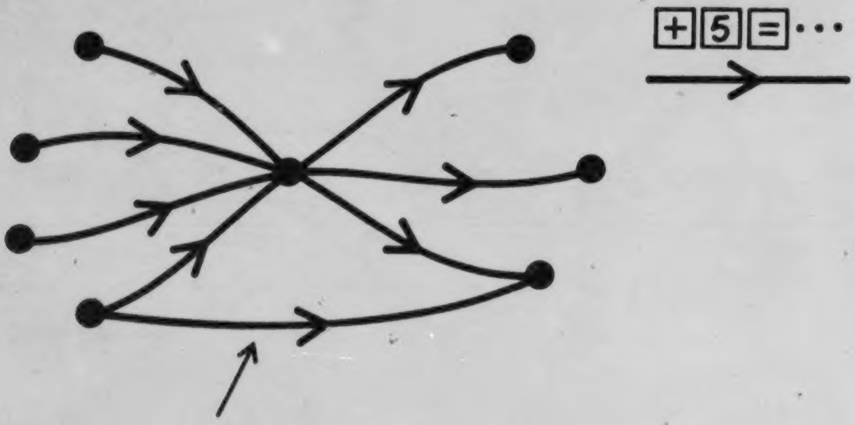
T: Do you notice a pattern to the numbers that appear?

S: The last digit is always 3 or 8.

Now erase the numerals and the blue return arrow from your arrow picture.

T: Some red arrows are missing from our picture. Who would like to trace a red arrow that we could add?

There are many possibilities. The example below is one that might occur in your class.

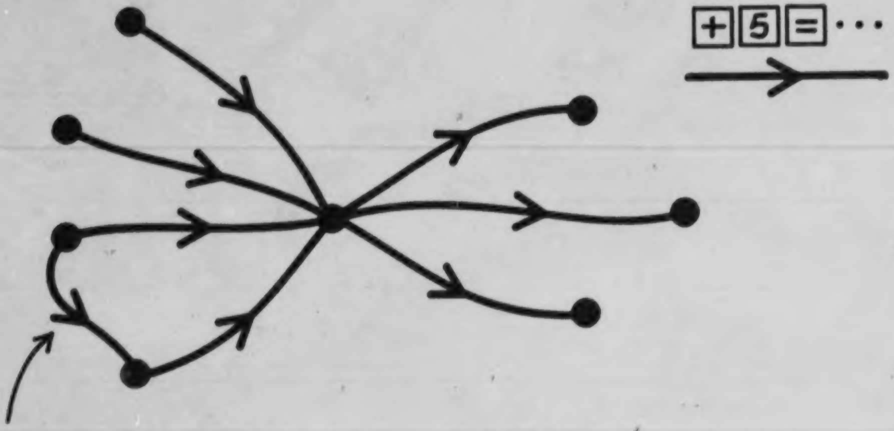


T: How do you know that we can draw this arrow?

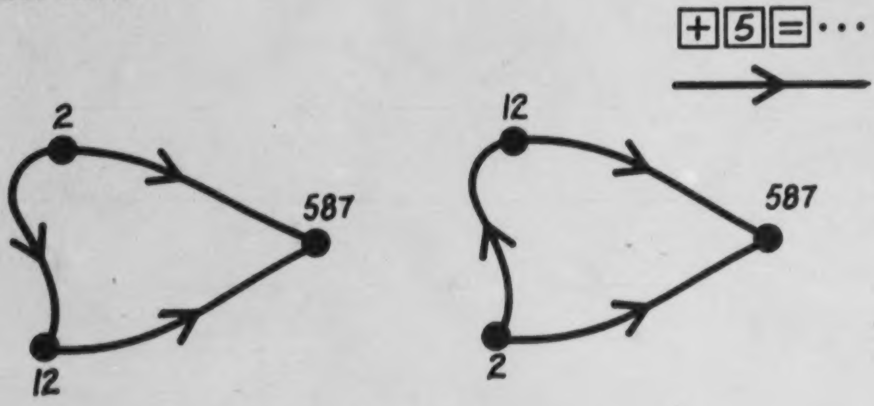
S: I can start with a number and press $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ $\boxed{=}$ and so on and get a second number. Then if I press $\boxed{=}$ some more I can get a third number. So I can draw an arrow from the first to the third number.

Accept any reasonable explanation.

NOTE: It is possible that a student will suggest drawing the following arrow.



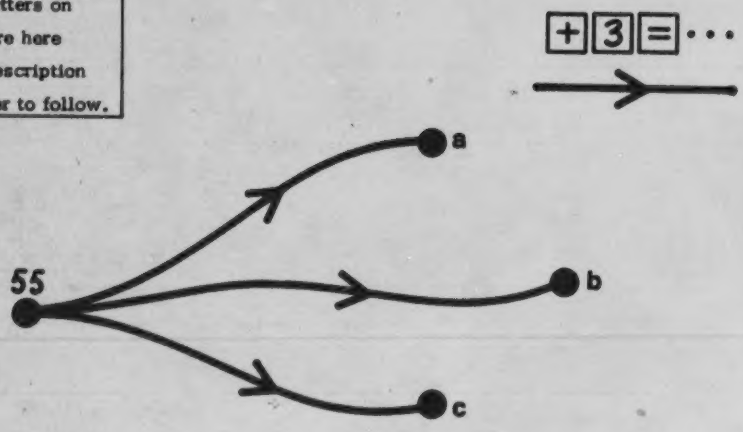
In fact an arrow may be drawn between any two beginning points or ending points; however, we are not sure of the direction of the arrow. Consider the two cases below.



Exercise 3

Draw this arrow picture.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



T: What numbers could we put in our arrow picture?

Let several students respond to this question. Label a and b with answers that occur during your discussion. Then point to c.

T: Can you name a number larger than 300?

S: 355.

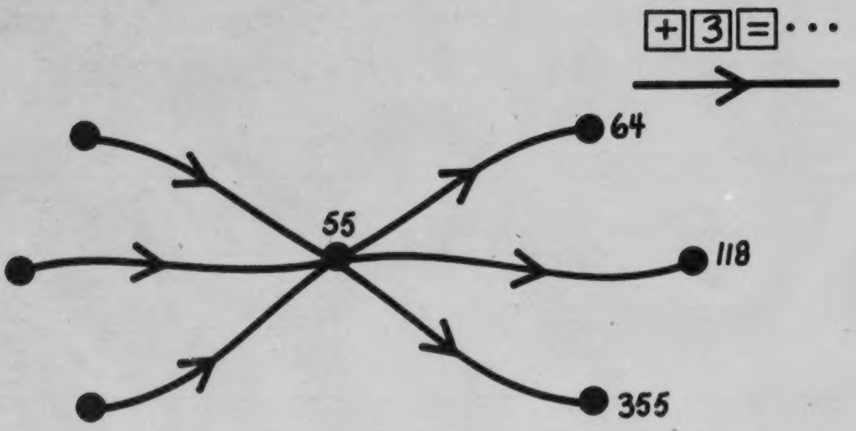
T: How much larger is 355 than 55?

S: 300.

T: Is 300 a multiple of 3?

S: Yes, 3 × 100 = 300.

Add these arrows to your drawing.



T: What could these dots be for?

Have the students give some of the answers they have found using their calculators. Label two of the dots with answers that you receive from the class. For the third dot, ask:

T: What is the smallest positive number?

S: 1.

T: What is the largest possible negative number that could be in our picture? (Answer: $^{-}2$)

If there are still those who have difficulty with this situation, suggest looking at return arrows.

T: The smallest positive number is 1. What could the return arrow be for?

S: $\boxed{-}$ $\boxed{3}$ $\boxed{=}$ $\boxed{=}$ and so on.

21 N

T: 1 - 3 = . . . ?

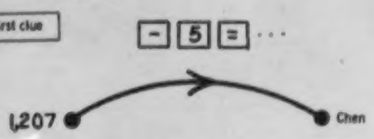
S: $^{-}2$.

Worksheets N69, N70, and N71 are available.

Name ____________ **N 69 ***

Chen is a secret number.

First clue



Chen could be 1,192 ; 457 ; 12 ; $^{-}28$; $^{-}1,008$ and so on.

What is the smallest positive number that Chen could be ? 2

Second clue (Chen can be any positive integer whose ones' digit is 2 or 7 or any negative integer whose ones' digit is 3 or 8.)

Chen can be put on this Minicomputer with exactly three positive checkers.

Chen is 7 or 12 or 17 .

Third clue

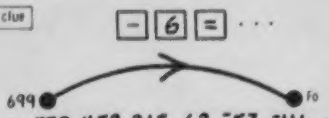
If you add 8 to Chen, you will get a square number.

Chen is 17 .

Name ____________ **N 70 ****

Fo is a secret number.

First clue



Fo could be 579 ; 453 ; 315 ; 63 ; $^{-}57$; $^{-}111$; and so on.

What is the smallest positive number that Fo could be ? 3

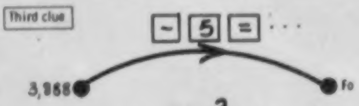
What is the largest negative number that Fo could be ? $^{-}3$

Second clue

Fo can be put on this Minicomputer with exactly three checkers (positive or negative).

Fo is 3 or 9 or 15 or $^{-}3$ or $^{-}9$ or $^{-}15$.

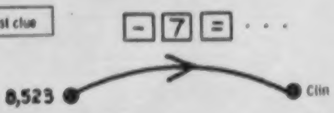
Third clue



Fo is 3 .

Name ____________ **N 71 *****

First clue



What is the smallest positive number that Clin could be ? 4

What is the largest negative number that Clin could be ? $^{-}3$

Second clue

Clin is a positive number smaller than 100.

Clin cannot be put on the Minicomputer with exactly three checkers (positive or negative).

Clin is 53 or 67 .

Third clue

If you add 20 to Clin, you will find a prime number.

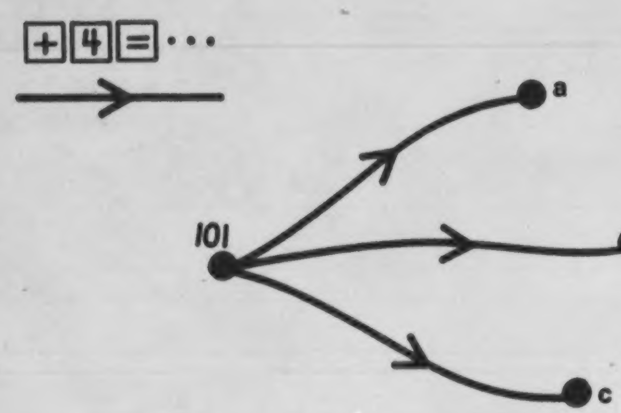
Clin is 53.

7.3 Hand-Calculator Relations #2

Distribute as many hand-calculators as you have available. If there are only a few calculators, begin the lesson with a warm-up activity similar to Section 7.2, Exercise 1.

Exercise 1

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: The red arrow is for $\boxed{+}$ $\boxed{4}$ $\boxed{=}$ $\boxed{=}$ and so on, as many times as you want. (Trace the arrow from 101 to a.) For this arrow, we will start with 101 and press $\boxed{+}$ $\boxed{4}$ $\boxed{=}$ $\boxed{=}$ and so on. What number could be here (point to a)?

Label a, b, and c with correct answers that students volunteer; e.g., 105; 125; 161; etc. Erase the numerals at a, b, and c and continue the exercise.

T: Label this dot (point to a) with the closest neighbor to 200.

S: 201.

It may happen that students will suggest 199. Remind them that if one starts from 101 and presses $\boxed{+}$ $\boxed{4}$ $\boxed{=}$ . . . , 199 will not appear on the display; 197 and 201 will.

T: Good. How many times would I have to press $\boxed{=}$ to get 201 on the display?

S: 25 times.

T: I want to label this dot (point to b) with the largest possible number smaller than 1,000.

If some students have some difficulty with this situation, suggest:

T: Think of the closest neighbor to 1,000. Rather than pressing $\boxed{=}$ so many times, think about the situation. Remember we said that starting at 101 and pressing $\boxed{+}$ $\boxed{4}$ then $\boxed{=}$ twenty-five times gives us 201. What if I press $\boxed{=}$ twenty-five more times?

S: 301.

Write "201" and "301" on the board.

201
301

T: If I press $\boxed{=}$ twenty-five more times?

S: 401.

27 N

Continue in this way until most students recognize the pattern.

T: Now what is the closest neighbor to 1,000?

S: 1,001.

T: Good. What is the largest possible number smaller than 1,000?

S: 997 = 1,001 - 4.

Label b "997".

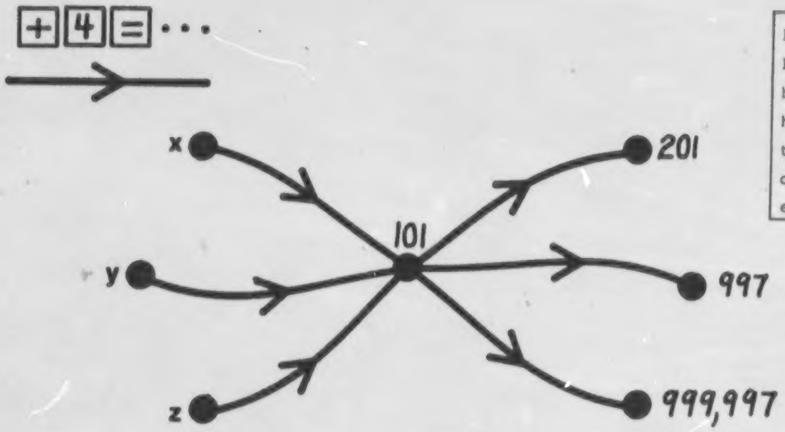
T: What is the closest neighbor to 1,000,000?

S: 1,000,001.

T: Write the largest possible number smaller than 1,000,000 on your paper.

S: 999,997.

Label c "999,997" and then add these arrows to your drawing.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

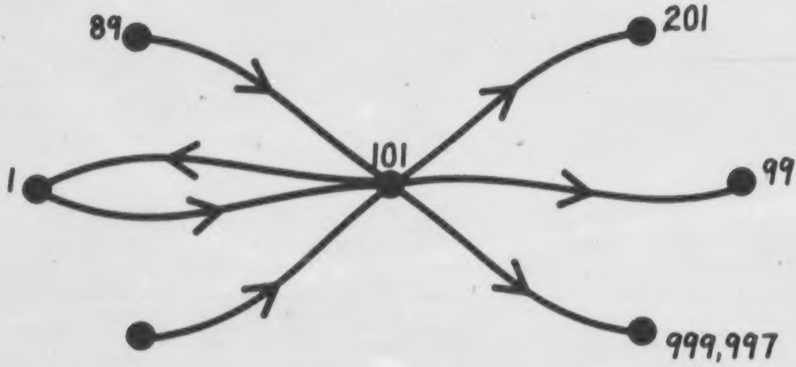
T: What number could be here (point to x)?

Let several students answer and label x with one correct answer; e.g., 97, 85, 57, etc.

T: Label this dot (point to y) with the smallest positive number it could be.

S: 1.

Label y. Add this blue arrow to your drawing.



T: What could this blue arrow be for?

S: $\boxed{-}$ $\boxed{4}$ $\boxed{=}$ $\boxed{=}$ ...

T: Good. How many times must I press $\boxed{=}$ to go from 101 to 1 on the display?

S: 25 times.

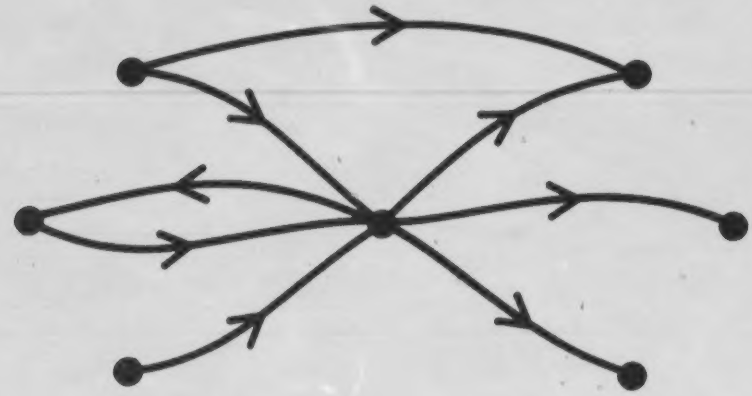
T: Label this dot (point to z) with the largest negative number it could be.

S: $^{-}3$.

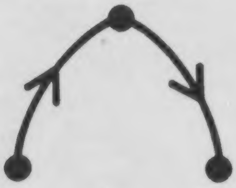
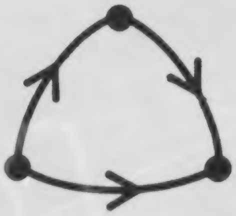
Erase the numerals from the arrow picture.

T: Can you trace some other red arrows that could be drawn in our picture?

There are many possibilities. Below we show one.

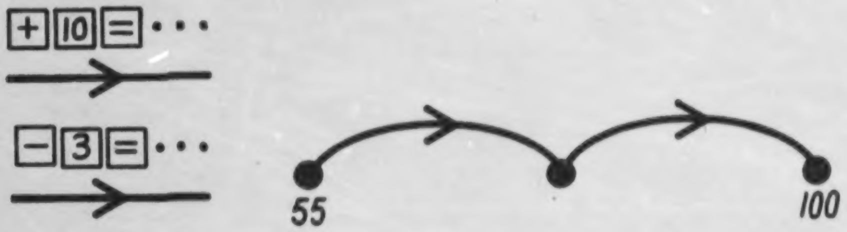


In general, if a red arrow is followed by a red arrow, then the red arrow starting where the first one started and ending where the second one ended can be drawn. That is,

if  then 

Exercise 2

Draw this picture on the board.



T: What number could be here (point to the middle dot)? There are many possibilities. Write at least one on your paper. Try to find a pattern.

Let the class work on this problem for a few minutes.

T: What is one number you found?

It will be helpful to make a list of the numbers on the board.

S: 115.

Start from 55 and trace the blue arrow.

T: How many times did you press $\boxed{=}$?

S: Six times.

Start from the middle dot and trace the red arrow.

T: How many times did you press $\boxed{=}$?

S: Five times.

T: Good. Who has another number?

S: 145.

Check that an answer is correct and then add it to your list.

T: Another?

S: 175.

Continue this until most students see a pattern.

T: Do you see a pattern?

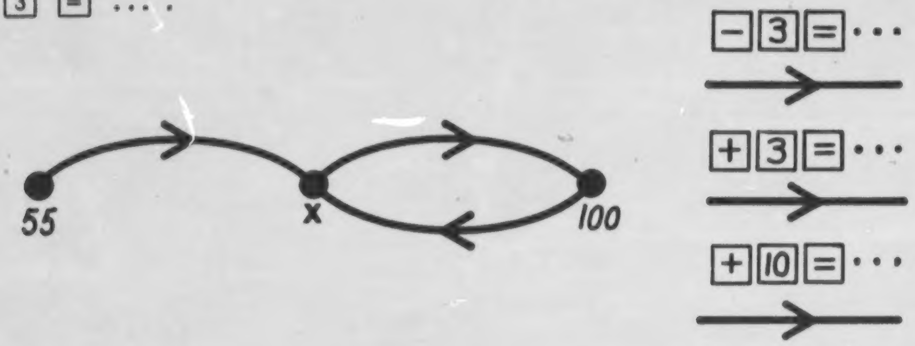
S: Each number is 30 more than the last.

T: Why is it 30 more?

At this point accept any reasonable explanation, because the idea may be difficult to verbalize. If no explanation is forthcoming, do not be concerned. You should not force an explanation at this early stage. A discussion of the +30 pattern is presented below for your information.

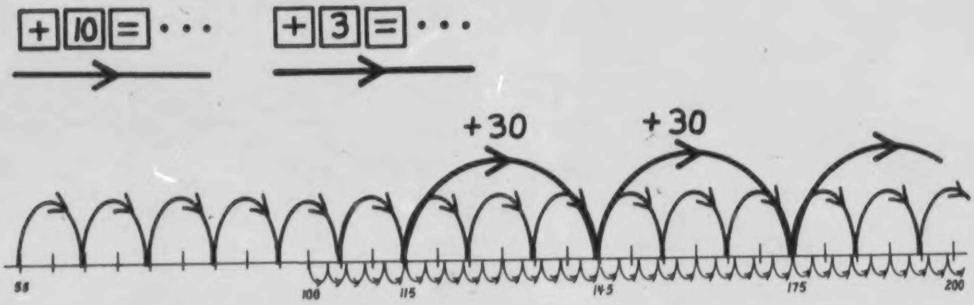
Consider the arrow picture with the addition of one of the return arrows,

$\boxed{+}\ \boxed{3}\ \boxed{=}$ . . . .

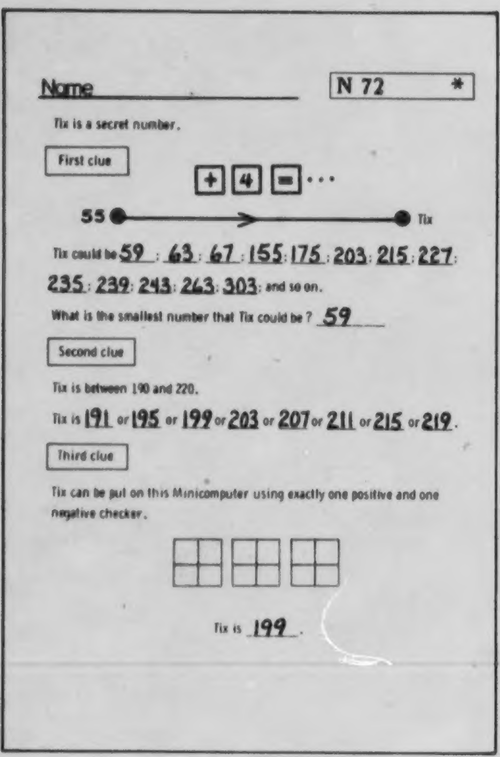


The blue arrow tells us that the number at x must end in 5 and the black arrow tells us that the number at x must also be 100 plus a multiple of 3. Thus x may be any number that ends with a 5 and is 100 plus a multiple of 3. A quick check shows that 115 is the smallest number that fulfills both conditions.

Now consider jumps on the number line.



Worksheets N72, N73, and N74 are available.



Name

N 72 *

Tix is a secret number.

First clue

+ 4 = ...

55 → Tix

Tix could be 59 ; 63 ; 67 ; 155 ; 175 ; 203 ; 215 ; 227 ; 235 ; 239 ; 243 ; 263 ; 303 ; and so on.

What is the smallest number that Tix could be ? 59

Second clue

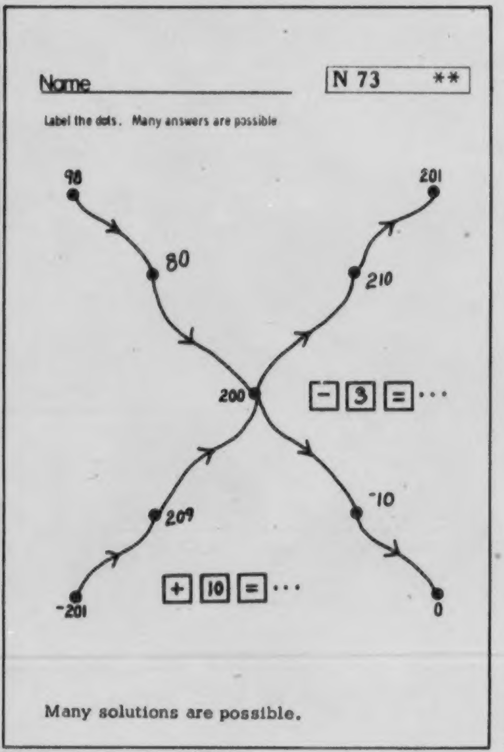
Tix is between 190 and 220.

Tix is 191 or 195 or 199 or 203 or 207 or 211 or 215 or 219.

Third clue

Tix can be put on this Minicomputer using exactly one positive and one negative checker.

Tix is 199.

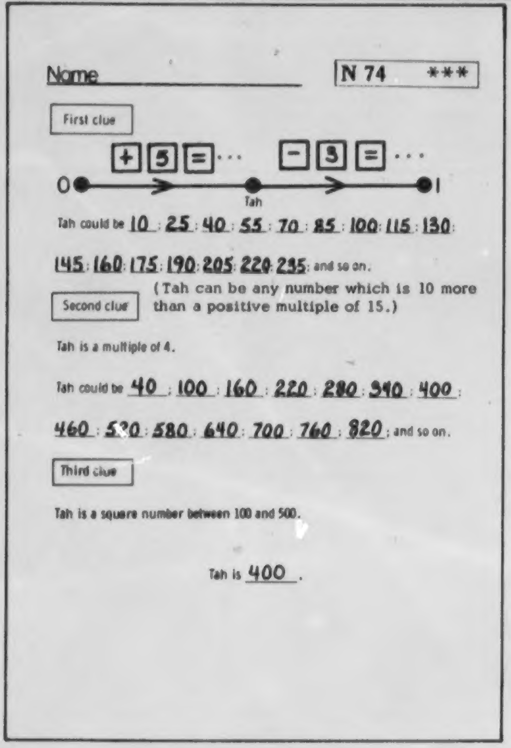


Name

N 73 **

Label the dots. Many answers are possible

Many solutions are possible.



Name

N 74 ***

First clue

+ 5 = ...   − 3 = ...

0 → Tah → 1

Tah could be 10 ; 25 ; 40 ; 55 ; 70 ; 85 ; 100 ; 115 ; 130 ; 145 ; 160 ; 175 ; 190 ; 205 ; 220 ; 235 ; and so on.

(Tah can be any number which is 10 more than a positive multiple of 15.)

Second clue

Tah is a multiple of 4.

Tah could be 40 ; 100 ; 160 ; 220 ; 280 ; 340 ; 400 ; 460 ; 520 ; 580 ; 640 ; 700 ; 760 ; 820 ; and so on.

Third clue

Tah is a square number between 100 and 500.

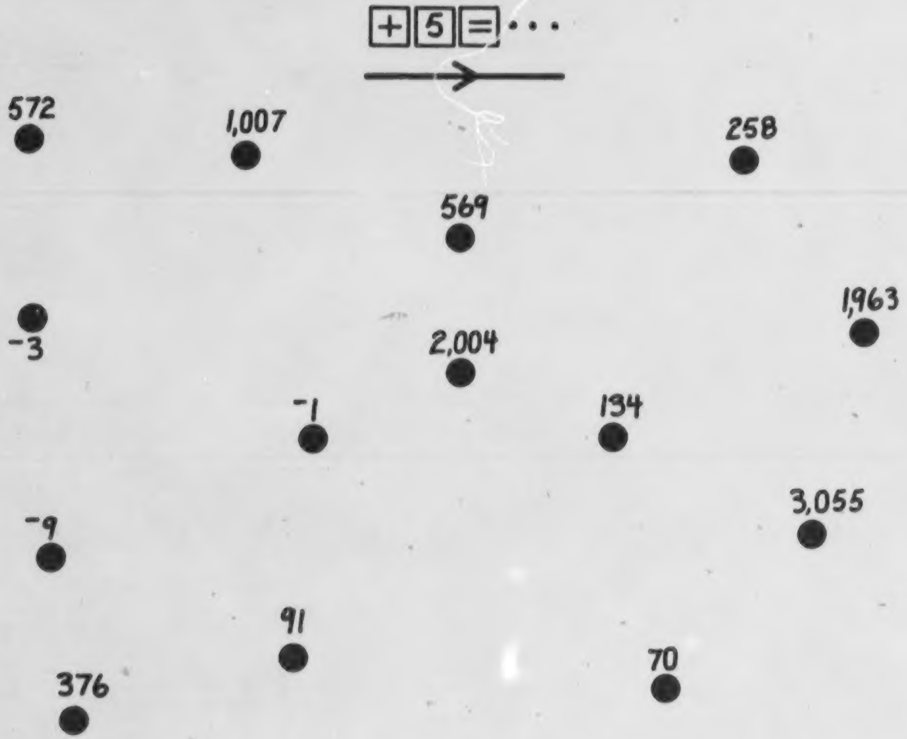
Tah is 400.

7.4 Hand-Calculator Relations #3

Distribute as many hand-calculators as you have available. If there are only a few calculators, begin the lesson with a warm-up activity similar to Section 7.2, Exercise 1.

Exercise 1

Draw this picture with carefully spaced dots.



NOTE: You might want to prepare this drawing ahead of time on the chalkboard or on an easel size sheet of paper which can be taped to the board. A careful placement of the dots will result in an interesting arrow picture that suggests many patterns.

T: Here is a picture with many dots; the red arrow is for $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ . . . . We are to draw as many red arrows as we can. Suppose we start at 572 and draw a red arrow from 572 to one of the other numbers. Put 572 on the display of your calculator and press $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ $\boxed{=}$ and so on, as many times as you like. Watch the numbers that appear on the display. What do you notice?

S: The last digit of each number that appears is either 2 or 7.

T: So, we can draw a red arrow from 572 to what number in our picture?

S: 1,007, because it ends with a 7.

Draw a red arrow from 572 to 1,007.

T: Where else can we draw red arrows?

Encourage the students to use the calculators to explore the situation. Have students come to the board and trace red arrows.

Suppose a student traces an arrow from 569 to 2,004.

T: Let's see if it is correct. Put 569 on the display of your calculators. Press $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ . . . . Will 2,004 appear?

S: Yes. The last digit is always 4 or 9.

T: So, we can draw an arrow from 569 to 2,004.

S: We can draw an arrow from ⁻9 to 91.

T: Let's see. Put $^{-}9$[*] on the display of your calculator and press $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ . . . . What numbers appear?

S: $^{-}4$.

S: 1.

S: 6.

S: The pattern of the last digit changes. The last digit is 9 or 4 for a negative number and 1 or 6 for a positive number. So we can draw an arrow from $^{-}9$ to 91.

S: And an arrow from $^{-}9$ to 376.

S: And an arrow from 91 to 376.

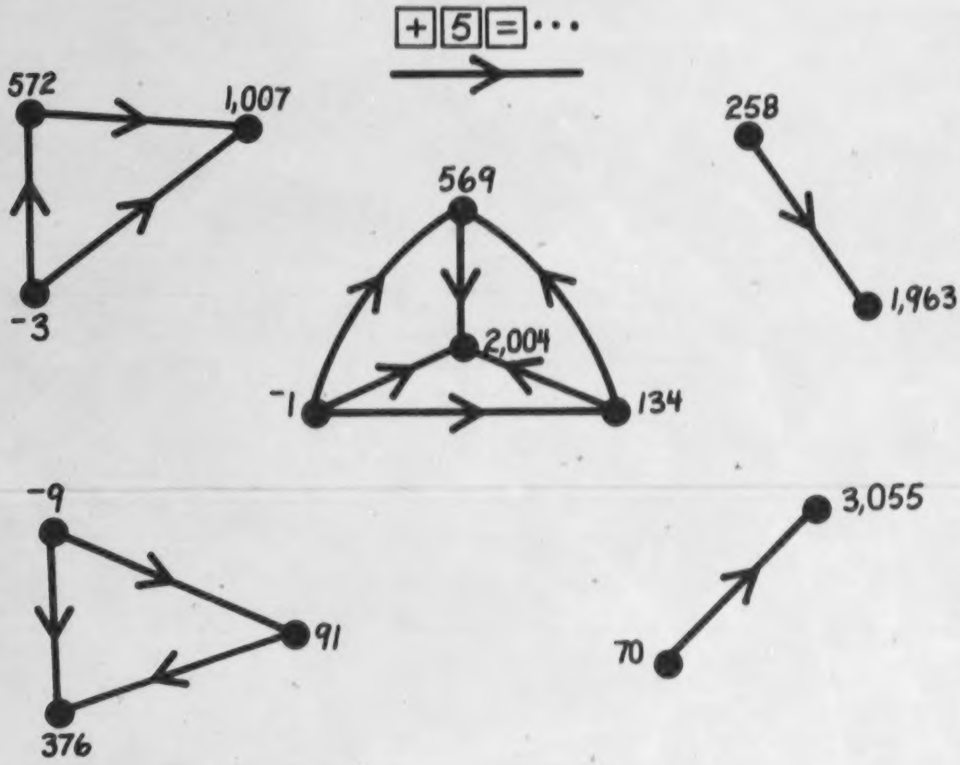
---

[*] Putting $^{-}9$ on the display of your calculator can be accomplished in one of at least two ways:

- Press $\boxed{0}$ $\boxed{-}$ $\boxed{9}$ $\boxed{=}$.
- If your calculator has a $\boxed{\text{CS}}$ key, press $\boxed{9}$ $\boxed{\text{CS}}$ or $\boxed{\text{CS}}$ $\boxed{9}$. Some calculators have a $\boxed{+/-}$ key instead of a $\boxed{\text{CS}}$ key.

33 N

Add these arrows to your picture. Continue this activity until the picture is complete. A careful drawing of the dots should yield this arrow picture.



T: Do you notice anything interesting about this arrow picture?

S: There are five separate pieces to the picture.

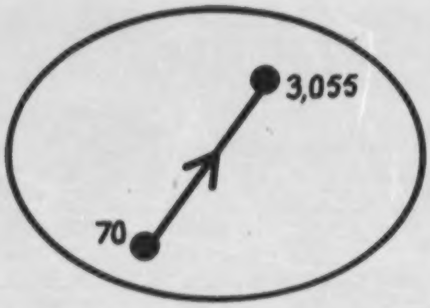
T: Why do you think there are five pieces?

S: Because of the five pairs of last digits: 0 and 5; 1 and 6; 2 and 7; 3 and 8; 4 and 9.

S: Because $\boxed{+}\ \boxed{5}\ \boxed{=}$ ... is like counting by fives.

Accept any reasonable answer.

Draw a string around this part of your picture.



T:	If I draw a string around this part of the picture, what other numbers could go inside the string?
S:	80.
S:	4,135.
T:	What is the smallest positive number that could go inside of the string?
S:	5.

If a student answers 0, point out that 0 would be inside the string but is not positive.

T:	The largest negative number?
S:	⁻5.
T:	Some other negative numbers?

S: $^-10$.

S: $^-25$.

S: $^-40$.

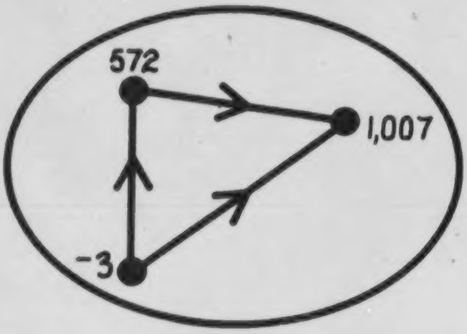
T: What do we call the numbers that are inside this string?

S: Multiples of 5.

T: Good. Do you see any patterns?

S: The numbers end in 0 or 5.

Now draw a string around this piece of your picture.



T: What other numbers could go inside this string?

S: 577.

S: 1,012.

T: What is the smallest positive number that could go inside the string?

S: 2.

T: The largest negative number?

S: ⁻3.

T: Some other negative numbers?

S: ⁻8.

S: ⁻13.

T: Do you see any patterns?

S: The negative numbers end in 3 or 8.

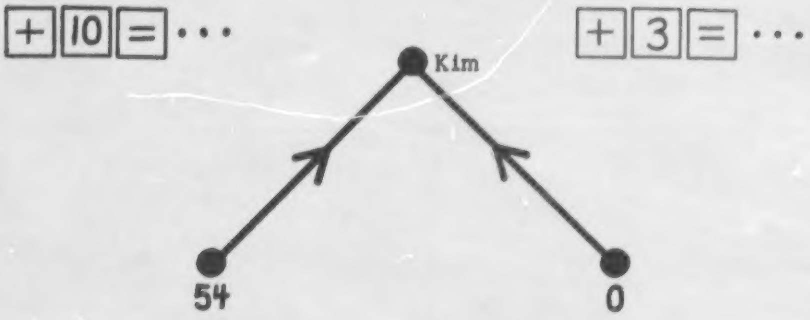
S: The positive numbers end in 2 or 7.

Some student might note that these numbers are 2 plus multiples of 5; e.g., 572 = 570 + 2; 1,007 = 1,005 + 2; 577 = 575 + 2; ⁻8 = ⁻10 + 2; etc. This would be a very good observation; but if it is not mentioned, you need not bring it to the class' attention at this time.

If there is interest, the other three pieces of the string picture can be investigated in a similar manner. However, extension of this activity is not necessary, especially if you are pressed for time.

Exercise 2

Draw this arrow picture on the board.



T: Kim is a secret number.

First clue

Kim is in this arrow picture. If you put 54 on your calculator and press $\boxed{+}$ $\boxed{10}$ $\boxed{=}$ ... , you will get Kim. Also if you put 0 on your calculator and press $\boxed{+}$ $\boxed{3}$ $\boxed{=}$ ... , you get Kim. Write on your paper some numbers that Kim could be.

Let the students have some time to explore the situation.

T: Who can tell me a number that Kim could be?

S: 84.

S: 114.

If there are some students who have difficulty with this clue, try this:

Suppose 64 is suggested as a number that Kim could be.

T: If I put 54 on the display of the calculator and press $\boxed{+}$ $\boxed{10}$ $\boxed{=}$ . . . , I will get 64. But if I put 0 on the calculator and press $\boxed{+}$ $\boxed{3}$ $\boxed{=}$ . . . , will I get 64, too?

S: No. You will get 60, 63, and 66 but not 64.

T: So Kim cannot be 64. What is the smallest number Kim could be?

S: 84.

T: Another number that Kim could be?

As the students make suggestions, record them on the board.

S: 114.

S: 144.

S: 174.

T: Can you give me more numbers without using your calculator?

S: Just add 30 each time.

S: 204.

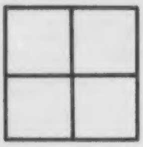
S: 234.

The students should conclude that Kim could be 84; 114; 144; 174; 204; 234; and so on.

Second clue

Display one Minicomputer board and these checkers.

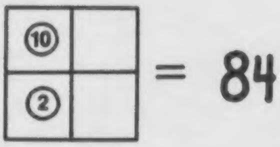
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨ ⑩



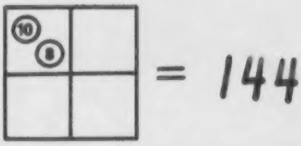
T: Kim can be put on the ones' board using this ⑩-checker and exactly one of these checkers (point to the checkers: ② - ⑨).

S: Kim could be 84.

T: Show us.



S: Kim could be 144.



T: Could Kim be 114?

S: No.

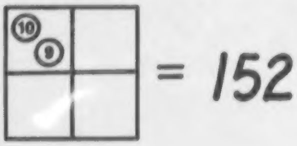
T: Why?

S: If I put the ⑩-checker on the 8-square, that would be 80. And I would need 34 more because 80 + 34 = 114. But 34 cannot be put on the Minicomputer with exactly one checker.

S: You could put the ⑩-checker on the 4-square, but then you cannot put 74 on the Minicomputer with exactly one checker.

The argument can be extended to include the cases of the ⑩-checker on the 2-square and the ⑩-checker on the 1-square. In each case a number larger than 72 is needed to reach 114, but 72 (the ⑨-checker on the 8-square) is the largest possible number using exactly one of the given checkers on the ones' board. Hence, we are sure that Kim cannot be 114.

We can argue that Kim cannot be larger than 144 by noting that the largest number that can be put on the Minicomputer with a ⑩-checker and exactly one other checker is 152.



Since 174 is the next number larger than 144 that Kim could be, it is clear that 174 or any larger number cannot be Kim.

It is not necessary to go into such a detailed explanation, but some students may find this argument interesting.

The class should conclude that Kim is 84 or 144.

Third clue

T: Kim is a square number. Can you give me an example of a square number?

S: 25, because 5 × 5 = 25.

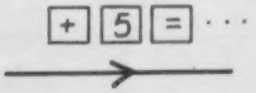
S: 9, because 3 × 3 = 9.

S: Kim is 144, because 12 × 12 = 144.

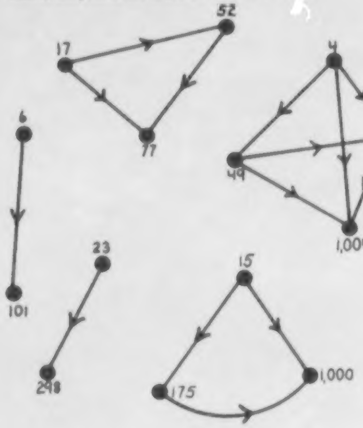
The students should conclude that Kim is 144.

Worksheets N75, N76, N77, and N78 are available.

Name ______________ **N 75 ***

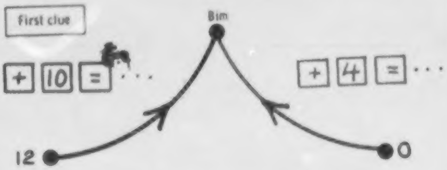


Draw as many red arrows as you can between these dots.


52
17
77
4
6
99
49
1,004
101
23
15
248
1,000
175

Name ______________ **N 76 ****

First clue



Bim could be 32 ; 52 ; 72 ; 92 ; 112 ; 132 ; 152 ; 172 ;

and so on. (Bim can be any number which is 32 more than a positive multiple of 20.)

Second clue

Bim can be put on this Minicomputer using exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

Bim is 32 or 72.

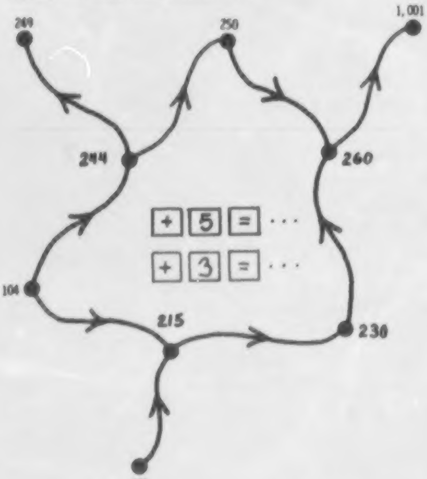
Third clue

Bim has exactly six positive divisors.

Bim is 32.

Name ______________ **N 77 *****

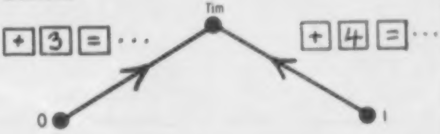
Label the dots. Many answers are possible.


249
250
1,001
244
260
+
5
=
. . .
+
3
=
. . .
104
215
230
205

Many solutions are possible.

Name ______________ **N 78 ******

First clue



Tim could be 9 ; 21 ; 33 ; 45 ; 57 ; 69 ; 81 ; 93 ;

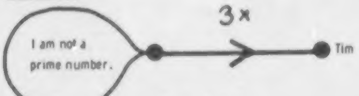
105 ; and so on. (Tim can be any number which is nine more than a positive multiple of 12.)

Second clue

Tim is between 200 and 240.

Tim is 201 or 213 or 225 or 237.

Third clue


3x
I am not a prime number.
Tim

Tim is 225.

8
DIVISION

The lessons in this chapter reinforce and continue the experiences of Part I of the CSMP Intermediate Grades Program. In these lessons the first stages of a division algorithm are set within the framework of familar situations. The use of arrow diagrams as tools for solving division problems is encouraged, as is estimation both for narrowing in on a solution and for checking whether or not a solution is reasonable. The relationship between multiplication and division is used explicitly to both solve problems and check solutions.

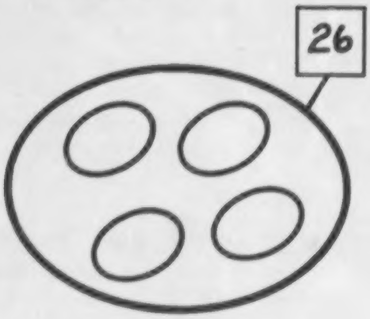
CAPSULE LESSON SUMMARIES

Section 8.1:	Solve division problems presented in several real situations and write both the division and related multiplication calculations. Present a road building problem and use the result to solve a division problem. (Worksheets N79, N80)
Section 8.2:	Label the arrows of an arrow picture with multiplication functions and use the arrow picture to help solve division problems. Explore several similar situations using arrow diagrams. (Worksheets N81, N82, N83)

8.1 Division #1

Exercise 1

Draw this string picture on the board.



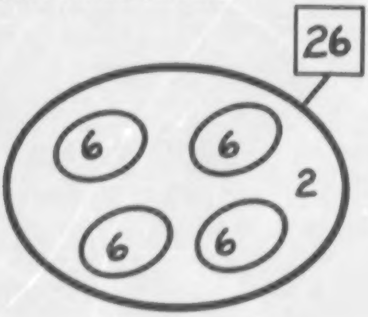
T: There are exactly 26 ping-pong paddles inside this string. I want to put the same number in each of the red strings and put as many as possible in each string. How could we do it?

S: Put five paddles in each red string and have six left over.

T: It's true that each string has the same number of paddles, but have you put as many as possible in each red string?

S: No. I can put six in each red string and have two left over.

Write this information in your string picture.



T: This is one calculation for our picture.

Write this calculation on the board.

$$(4 \times 6) + 2 = 26$$

T: Can anyone think of another calculation for this picture?

Accept any answer that is correct but emphasize this calculation.

$$\begin{array}{r} 6 \\ 4\overline{)26} \end{array} \quad R = 2^{*}$$

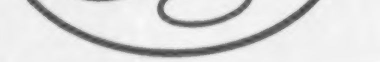
Change the label to "103" and repeat the exercise. Stress the two calculations:

$$(4 \times 25) + 3 = 103 \quad \& \quad \begin{array}{r} 25 \\ 4\overline{)103} \end{array} \; R = 3$$

Repeat the exercise using 405.

$$(4 \times 101) + 1 = 405 \quad \& \quad \begin{array}{r} 101 \\ 4\overline{)405} \end{array} \; R = 1$$

Draw this string picture on the board.



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[*] Read this as "26 divided by 4 is 6 with a remainder of 2."

T: We have a new situation with exactly 76 ping-pong paddles. Put the same number in each of the red strings and put as many as possible in each string. How should we do it?

S: 15 in each string with one left over.

T: Can someone give me a calculation?

Write the calculations on the board.

$$(5 \times 15) + 1 = 76 \qquad \begin{array}{r} 15 \;\; R = 1 \\ 5\,\overline{)76} \end{array}$$

Next change the label to "319". If some of the students have difficulty in getting started, ask:

T: Who can give me an estimate for the number that should go in each red string?

S: 60, because 5 × 60 = 300.

T: That's a good estimate. If we were dividing 300 ping-pong paddles, there would be 60 paddles in each of the five strings. But our problem is about 319 paddles.

S: 5 × 63 = 315 and there are four paddles left over.

T: Very good. Who can write a calculation for this problem?

Let a student write the calculations on the board.

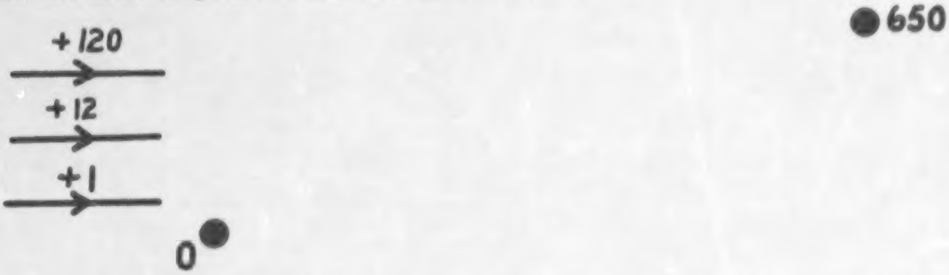
$$(5 \times 63) + 4 = 319 \qquad \begin{array}{r} 63 \;\; R = 4 \\ 5\,\overline{)319} \end{array}$$

Change the label from "319" to "1,001" and try the exercise again.

$$(5 \times 200) + 1 = 1{,}001 \qquad \begin{array}{r} 200 \;\; R=1 \\ 5\,\overline{)1{,}001} \end{array}$$

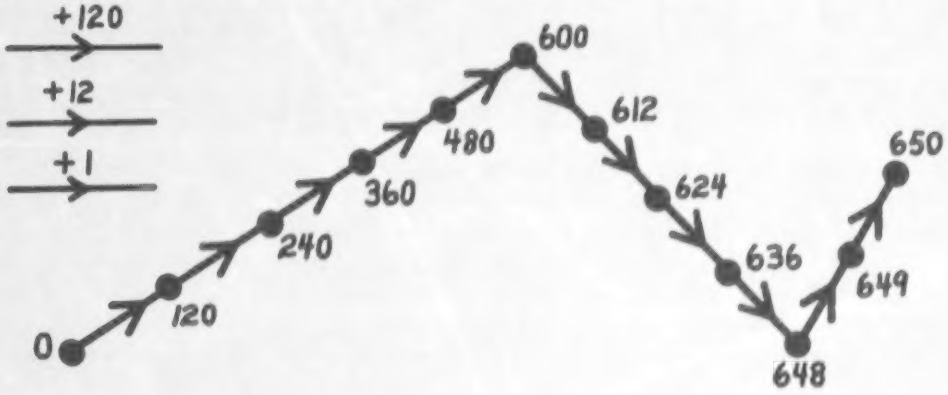
Exercise 2

Draw the following arrows and dots on the board.[*]



T: Build a blue-red-black road from 0 to 650 using as few arrows as possible.

Give the students several minutes to work on this problem. Then let a student who has drawn a road with eleven arrows put it on the board.



---

[*] In this exercise, the black arrows may be replaced by any color other than red or blue. You may use white or some other color on the blackboard, but in this guide it is simplest for us to describe the exercise using black arrows.

19 N

Write this problem next to the arrow picture on the board.

650 eggs.
Each carton holds 12 eggs.
How many completely filled cartons? ______
How many eggs left over? ______

T: You can solve this problem with the help of the arrow picture. Write your answer on a sheet of paper.

When several students have solved the problem (54 cartons are completely filled and 2 eggs are left), point to the first blue arrow and its ending dot.

T: How many cartons are needed to hold 120 eggs?

S: 10 cartons, because 10 × 12 = 120.

T: Each time there is a blue arrow, it means 10 cartons are filled. With five blue arrows, how many cartons are filled?

S: 50 cartons.

T: How many eggs are there altogether in the 50 cartons?

S: 600 eggs, because 50 × 12 = 600.

Point to the four red arrows.

T: From 600 to 648, how many more eggs?

S: 48 eggs.

T: How many cartons?

S: 4 cartons.

T: How many cartons are filled altogether?

S: 54 cartons.

T: How many eggs left over?

S: 2 eggs.

T: Can you write a calculation for this problem?

Have a student write the calculations on the board.

$$(12 \times 54) + 2 = 650 \qquad \begin{array}{r} 54 \\ 12\overline{)650} \end{array} \; R = 2$$

Worksheets N79 and N80 are available for the rest of the lesson. Should your faster workers finish early, write these three-star problems on the board for them.

$\boxed{***}$

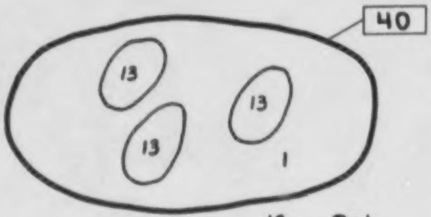
Note: Answers are in the boxes.

Use an arrow road to solve each problem.

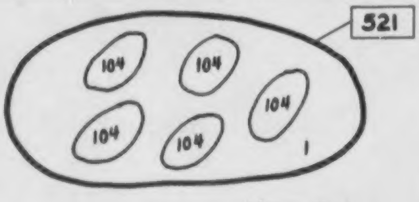
1,300 doughnuts. Each box holds 15 doughnuts.
How many completely filled boxes? $\boxed{86}$
How many doughnuts left over? $\boxed{10}$

4,960 sticks of gum. Each package holds 75 sticks.
How many completely filled packages? $\boxed{66}$
How many sticks of gum left over? $\boxed{10}$

Name ______________________ **N 79** *



Calculation: $(3\times13)+1=40$ or $\begin{array}{r}13\\3\overline{)40}\end{array}$ R=1

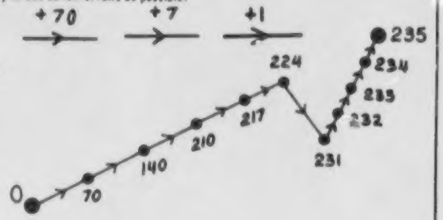

521
104
104
104
104
104
104
1

Calculation: $(5\times104)+1=521$ or $\begin{array}{r}104\\5\overline{)521}\end{array}$ R=1

Name ______________________ **N 80** **

Build a blue-red-black road from 0 to 235.

Try to use as few arrows as possible.


+70
+7
+1
0
70
140
210
217
224
231
232
233
234
235

235 tennis balls
Each can contains 7 tennis balls.

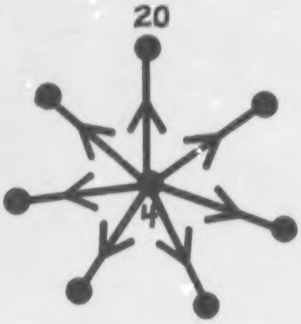
How many completely filled cans ? 33

How many tennis balls left over ? 4

8.2 Division #2

Exercise 1

Draw this arrow picture on the board.

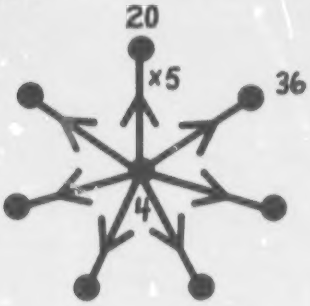


T (tracing the arrow ending at 20): I would like to label this arrow

× some number.

S: × 5.

Label the arrow "× 5". Move clockwise to the next dot and label it "36".



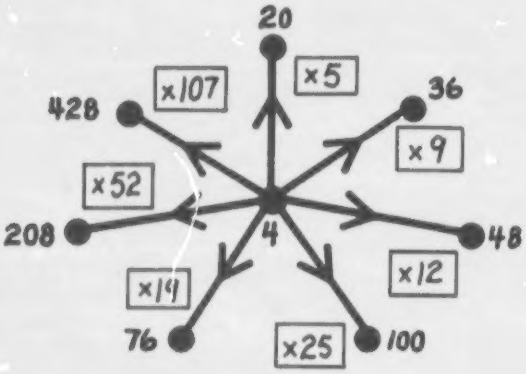
T (tracing the arrow ending at 36): What could this arrow be for? Again,

I would like it to be × some number.

S: × 9.

Label the arrow "× 9". Move clockwise and label the next dot "48".

Continue in this manner until the picture has been completed as shown below. Correct labels for the arrows supplied by students are in the boxes.



If there are some students who have difficulty with the multiplication, encourage them to first estimate a product. For example, you might suggest:

T: Instead of thinking about 76, think of 80.

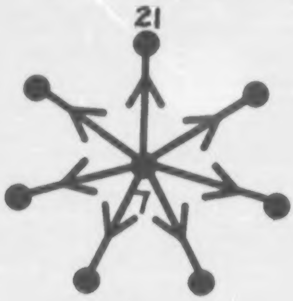
$4 \times \square = 80$?

S: $4 \times 20 = 80$.

T: So $76 = 4 \times \square$?

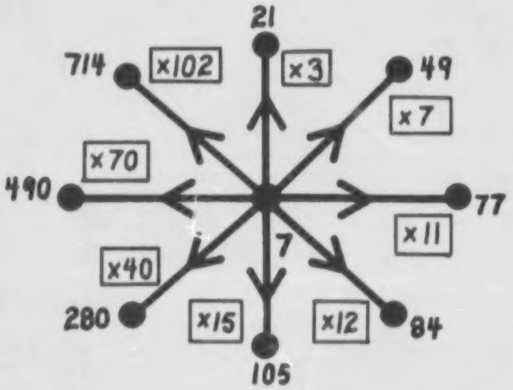
S: $76 = 4 \times 19$.

Repeat the exercise using this arrow picture.



Again, each arrow will be labeled " $\times\square$".

Work clockwise around the picture. A completed picture with labels for the arrows to be supplied by students in boxes is shown below.



For the arrow ending at 280, you could extend your questioning.

T: What could this arrow be for?

S: × 40.

T: If I changed "280" to "2,800" (change this in your drawing), what could the arrow be for?

S: × 400.

If a multiplication problem is too difficult at first, encourage estimation. For example,

T: What could this arrow be for (trace the arrow ending at 714)? Can you give an estimate?

S: 7 × 100 = 700.

T: Good. 714 is just how many more seven's?

S: 2.

S: $7 \times 102 = 714$.

Write these problems next to your arrow picture.

$105 \div 15 = \_\_\_\_$ $714 \div 7 = \_\_\_\_$

$280 \div 7 = \_\_\_\_$

T: Copy each problem on your paper. Use the arrow picture to help solve the problems.

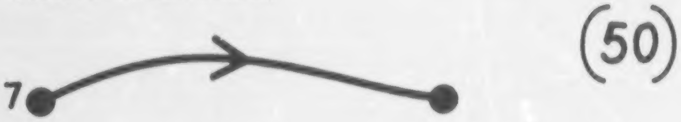
Let the students work individually for a few minutes, then check the work collectively.

$105 \div 15 = \underline{7}$ $714 \div 7 = \underline{102}$

$280 \div 7 = \underline{40}$

Exercise 2

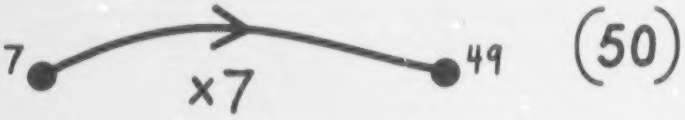
Draw this arrow picture on the board.



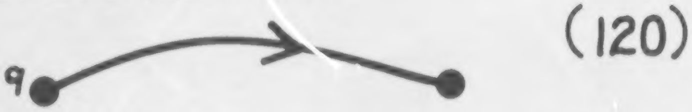
T: Label this arrow × some number so that the number here (point to the dot on the right) is as close to 50 as possible.

S: ×7, because $7 \times 7 = 49$ and 49 is only 1 away from 50.

Label the arrow and the end point.



Erase all labels and label the picture as below.



Give the students an opportunity to label the arrow and the end point. If some students have difficulty with this situation, encourage them to estimate. For example, you might ask:

T: Would × 10 be a good estimate?

S: No, 9 × 10 = 90.

S: 9 × 11 = 99.

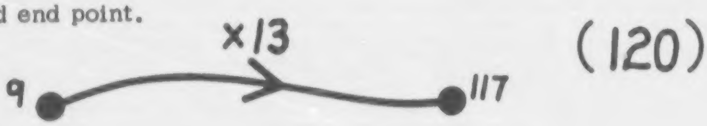
S: 9 × 12 = 108.

S: 9 × 13 = 117, just 3 away.

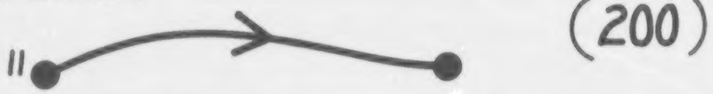
S: 9 × 14 = 126; 6 away.

T: So × 13 would be the best.

Label the arrow and end point.



Continue with a new situation.



T: What would be a good estimate?

S: × 20.

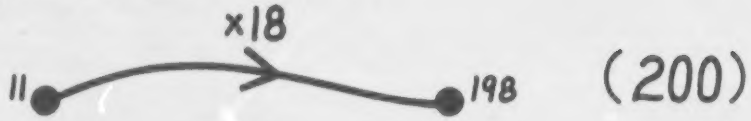
T: What is 11 × 2? (22) Then what is 11 × 20?

S: 220. It's too high.

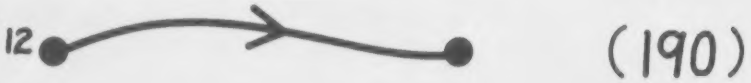
S: 11 × 19 = 209. Just subtract 11 from 220.

S: 11 × 18 = 198.

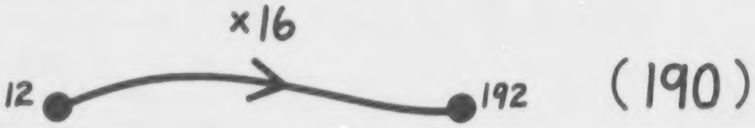
Label the arrow and end point.


×18
11
198
(200)

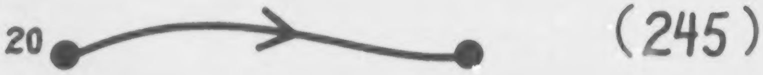
Repeat the exercise with this situation.



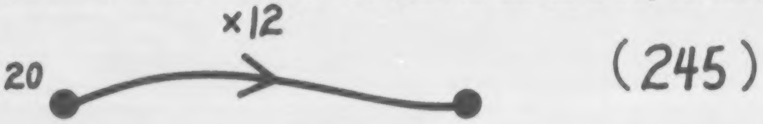
In this instance the closest number is larger than 190.


×16
12
192
(190)

Next, consider this situation.



Let the students label the arrow and the end point in the same way as above.



Then write this problem next to your arrow picture.

$$20\overline{)245}$$

T: Use the arrow picture to solve this problem.

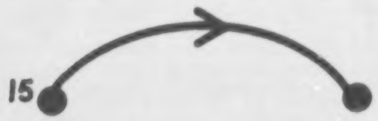
S: 245 ÷ 20 = 12 with 5 left over.

T: That's correct. We write it this way

$$\begin{array}{r} 12 \\ 20\,\overline{)245} \end{array} \quad R = 5$$

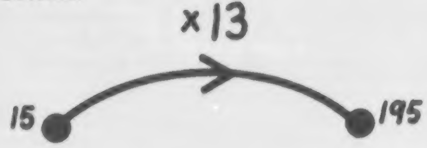
and read it as: 245 divided by 20 is 12 with a remainder of 5.

Change the situation and do a similar exercise.



$$(200) \qquad 15\,\overline{)200}$$

Solution:

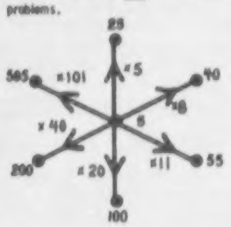


$$(200) \qquad \begin{array}{r} 13 \\ 15\,\overline{)200} \end{array} \quad R = 5$$

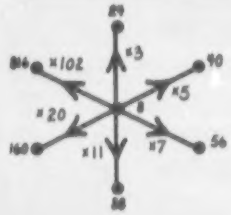
Worksheets N81, N82, and N83 are available.

Name____________________ N 81 *

Label each arrow "x ☐". Use the arrow pictures to solve the division problems.


25
505
×101
×5
40
×8
×40
5
200
×20
×11
55
100

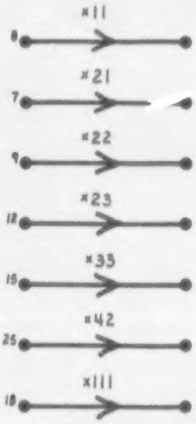
25 ÷ 5 = 5
40 ÷ 5 = 8
55 ÷ 5 = 11
100 ÷ 5 = 20
200 ÷ 5 = 40
505 ÷ 5 = 101


24
816
×102
×3
40
×5
×20
8
160
×11
×7
56
88

24 ÷ 8 = 3
40 ÷ 8 = 5
56 ÷ 8 = 7
88 ÷ 8 = 11
160 ÷ 8 = 20
816 ÷ 8 = 102

Name____________________ N 82 **

Label each arrow "x ☐". The ending number has to be the closest neighbor possible to the number in parentheses.
Use the arrow picture to solve the division problems.


×11
8
×21
7
×22
9
×23
12
×33
15
×42
25
×111
18

(90) $\begin{array}{r}11\\8\overline{)90}\end{array}$ r = 2

(150) $\begin{array}{r}21\\7\overline{)150}\end{array}$ r = 3

(200) $\begin{array}{r}22\\9\overline{)200}\end{array}$ r = 2

(280) $\begin{array}{r}23\\12\overline{)280}\end{array}$ r = 4

(500) $\begin{array}{r}33\\15\overline{)500}\end{array}$ r = 5

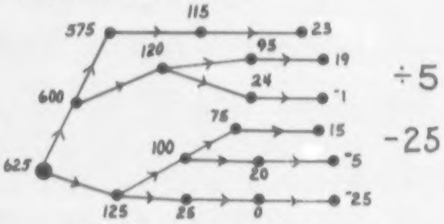
(1060) $\begin{array}{r}42\\25\overline{)1060}\end{array}$ r = 10

(2000) $\begin{array}{r}111\\18\overline{)2000}\end{array}$ r = 2

Name____________________ N 83 ***

First clue

Rank is the ending number of a red-blue road starting at 625 and using exactly two red arrows and two blue arrows.


115
575
23
120
95
19
÷5
24
600
⁻1
75
15
−25
100
625
⁻5
20
125
25
0
⁻25

Rank is 23 or 19 or 15 or ⁻1 or ⁻5 or ⁻25.

Second clue

If you add 510 to Rank, you will find a square number.

Rank is 19.

9

RATIONAL NUMBERS[*]

Making extensive use of arrow pictures, the lessons in this chapter involve the students with the functions $\frac{1}{2}\times$, $\frac{1}{3}\times$, $\frac{2}{3}\times$, $\frac{3}{5}\times$, etc. and compositions of these functions related to multiplication of fractions. In these lessons, the following properties of rational numbers and the above functions are explored explicitly.

a) $\frac{1}{3}\times$ is the same function as $\div 3$ and is the converse of $3\times$

b) $\frac{2}{3}\times$ is the composition of $2\times$ followed by $\div 3$ or of $\div 3$ followed by $2\times$

c) a rational number may have many fractional names.

CAPSULE LESSON SUMMARIES

Section 9.1:	Label dots and arrows in arrow pictures where rational numbers are involved and where the converse relationship between such functions as $3\times$ and $\frac{1}{3}\times$ is emphasized. Use arrows for the composition of two multiplication functions and for the converses of such functions to begin exploring multiplication of fractions. (Worksheets N84, N85, N86)
Section 9.2:	Use arrows for the composition of two functions to present functions such as $\frac{2}{3}\times$ as $2\times$ followed by $\frac{1}{3}\times$ and as $\frac{1}{3}\times$ followed by $2\times$. Continue to explore multiplication of fractions with the aid of arrows for the composition of two multiplication functions. (Worksheets N87, N88, N89)
Section 9.3:	Continue the activities of Section 9.2, again emphasizing that $\frac{1}{3}\times$ and $\div 3$ are the same function. Use arrow pictures to emphasize the fact that a rational number may have many fractional names. (Worksheets N90, N91, N92, N93)

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[*] The rational numbers include the integers and all numbers which have fractional names with integer numerators and positive integer denominators; e.g., $\frac{1}{2}$, $\frac{2}{3}$, $\frac{81}{734}$, $\frac{9}{5}$, $\frac{{}^-3}{4}$, $\frac{{}^-7}{3}$, and so on. Decimal numbers are also rational numbers since, for example, $0.5 = \frac{1}{2}$; $1.3 = \frac{13}{10}$; and so on.

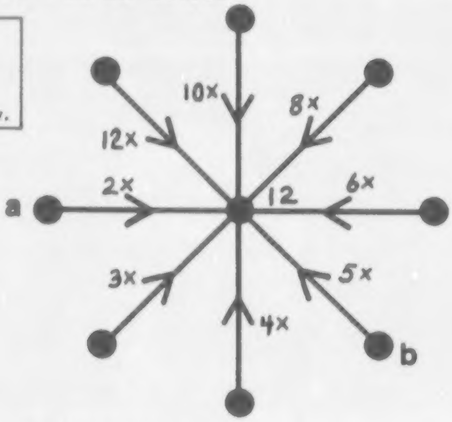
For communicating with the students, we suggest ignoring the distinction between rational numbers and their fractional names and refering to them simply as "fractions". We feel this is a better pedagogical practice at this level than using "rational numbers" even though it means accepting statements such as "0.5 is a fraction" for the time being.

9.1 Fractions #1

Exercise 1

Draw this arrow picture on the board.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



T: Together we will label the dots.

Start labeling at a and work counterclockwise. If the students have difficulty with the 5× arrow, ask:

T: Can this number be a whole number (point to b)?

S: No, the number is between 2 and 3.

T: Why?

S: 5 × 2 = 10 and 5 × 3 = 15, and 12 is between 10 and 15.

S: It is 2.2.

T: What is 5 × 2.2?

Write this problem on the board and work it with the class.

$$\begin{array}{r} \overset{1}{2}.2 \\ \times\ 5 \\ \hline 11.0 \end{array}$$

T: What is 5 × 0.2?

S: 1.0.

T: What is 5 × 0.3?

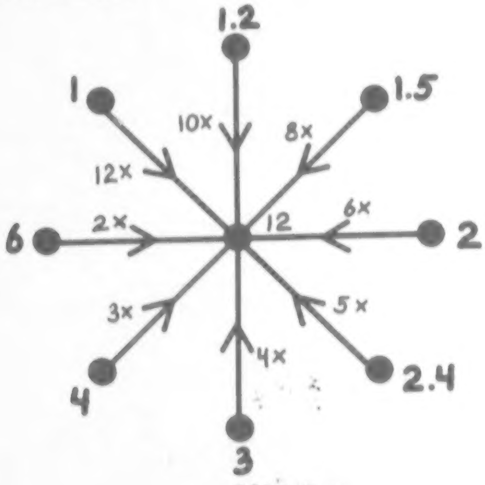
S: 1.5.

T: What is 5 × 0.4?

S: 2.

S: The missing number is 2.4, because 5 × 2.4 = 12.

Continue in this manner completing the arrow picture. For the more difficult problems, use a method similar to the above. Let the students first make an estimate, and then multiply to find how close they come to 12. A completed arrow picture is shown below.



Write this set of problems on the board.

$$\frac{1}{3} \times 12 = \square \qquad \frac{1}{6} \times 12 = \square$$

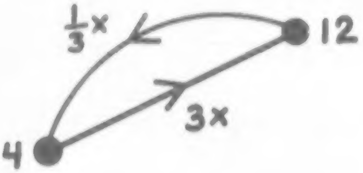
$$\frac{1}{4} \times 12 = \square \qquad \frac{1}{8} \times 12 = \square$$

T: Try to use the arrow picture to solve these problems. Write the answers on your paper.

After a minute or two, ask:

T: Who can trace an arrow for $\frac{1}{3}\times$?

Let a student trace the $\frac{1}{3}\times$ arrow starting at 12 and ending at 4.



T: $\frac{1}{3} \times 12 = 4$.

Continue this way for the three remaining problems.

$$\frac{1}{4} \times 12 = 3$$

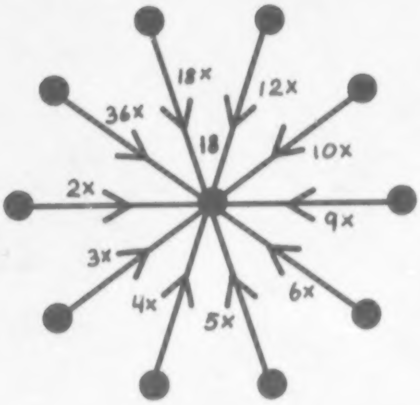
$$\frac{1}{6} \times 12 = 2$$

$$\frac{1}{8} \times 12 = 1.5$$

Exercise 2

T: This time the arrows will all point to 18. As I draw the picture on the board, you try to figure out the number at the start of each arrow.

Draw:


18×
12×
36×
18
10×
2×
9×
3×
6×
4×
5×

Continue as in Exercise 1. If the class has difficulty with, for example, the starting dot for the 12 × arrow, suggest:

T: 12 times some number equals 18. Is the number a whole number?

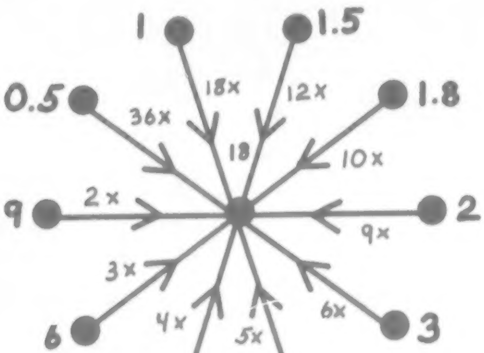
S: No, it is between 1 and 2.

T: What could the number be?

S: 1.5.

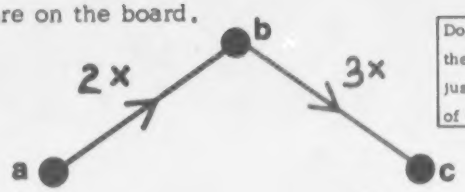
Check this by performing the calculation at the board in a way similar to the method used in Exercise 1.

A completed arrow picture is shown below.


1
1.5
0.5
18×
12×
1.8
36×
18
10×
9
2×
2
9×
3×
6×
4×
5×
6
3
4.5
3.6

Exercise 3

Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

Label the dot at c as "24".

T: What number is here (point to b)?

S: 8, because 3 × 8 = 24.

T: Good. What number is here (point to a)?

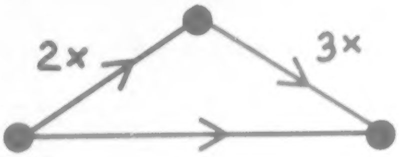
S: 4, because 2 × 4 = 8.

Repeat this situation several times more. Below are some suggestions for you to consider. The boxes indicate correct responses.

Feel free to adjust the situations to the numerical ability of your class.

a	b	c
$\boxed{3.5}$	$\boxed{7}$	21
$\boxed{12.5}$	$\boxed{25}$	75
$\boxed{1.25}$	$\boxed{2.5}$	7.5
$\boxed{25}$	$\boxed{50}$	150

Add this green arrow to your picture.

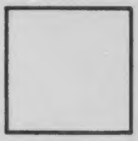


T: Write on your paper what you think the green arrow could be for.

Some students may label the green arrow "5 ×". The following discussion should help them see their mistake.

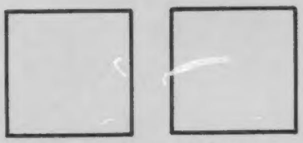
T: I put some dots on a sheet of paper. You cannot see the dots because they are invisible.

Draw this picture on the board.



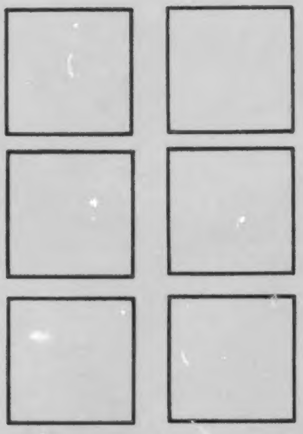
T: Then I double the number of dots.

Extend the picture.



T: Then I take 3 × this new number of dots.

Extend the picture.



T: Altogether I have 6 × the original number of dots.

Label the green arrow "6 ×".

20 N

Label the dot at a "25", at b "50", and at c "150".

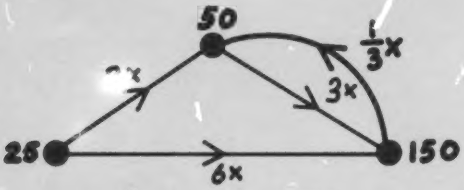
T: Use this arrow picture to solve these problems.

Write these three problems on the board.

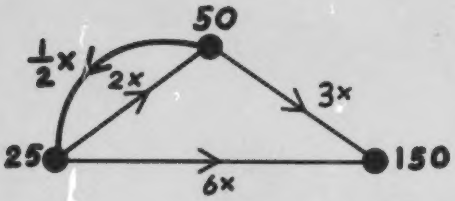
$\frac{1}{3} \times 150 =$ $\frac{1}{2} \times 50 =$ $\frac{1}{6} \times 150 =$

T: Who can trace a $\frac{1}{3}\times$ arrow and solve the first problem?

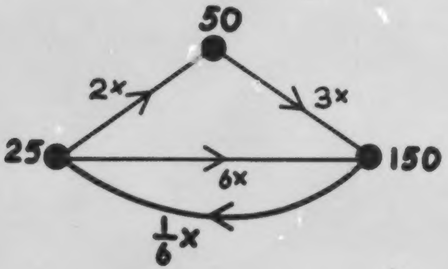
Let a student come up and trace the $\frac{1}{3}\times$ arrow. Stress that the arrow picture tells us that $\frac{1}{3} \times 150 = 50$, $\frac{1}{2} \times 50 = 25$, and $\frac{1}{6} \times 150 = 25$.


50
$\frac{1}{3}\times$
2×
3×
25
150
6×

$\frac{1}{3} \times 150 = 50$


50
$\frac{1}{2}\times$
2×
3×
25
150
6×

$\frac{1}{2} \times 50 = 25$

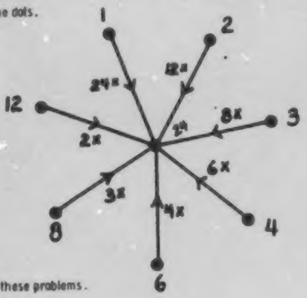

50
2×
3×
25
6×
150
$\frac{1}{6}\times$

$\frac{1}{6} \times 150 = 25$

Worksheets N84, N85, and N86 are available.

Name ____________________ N 84 *

Label the dots.


1
2
12
3
8
4
6
24x
12x
8x
2x
2x
6x
3x
4x
24

Solve these problems.

$\frac{1}{2}$ x 24 = 12

$\frac{1}{3}$ x 24 = 8

$\frac{1}{4}$ x 24 = 6

$\frac{1}{6}$ x 24 = 4

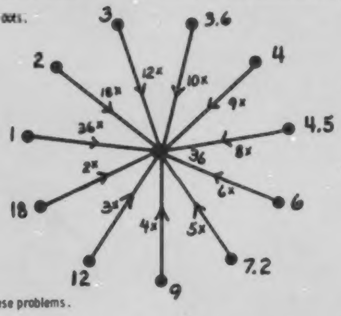
$\frac{1}{8}$ x 24 = 3

$\frac{1}{12}$ x 24 = 2

$\frac{1}{24}$ x 24 = 1

Name ____________________ N 85 **

Label the dots.


3
3.6
2
4
1
4.5
18
6
12
9
7.2
12x
10x
18x
9x
36x
8x
36
2x
6x
3x
4x
5x

Solve these problems.

$\frac{1}{2}$ x 36 = 18

$\frac{1}{3}$ x 36 = 12

$\frac{1}{4}$ x 36 = 9

$\frac{1}{5}$ x 36 = 7.2

$\frac{1}{6}$ x 36 = 6

$\frac{1}{8}$ x 36 = 4.5

$\frac{1}{9}$ x 36 = 4

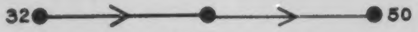
$\frac{1}{10}$ x 36 = 3.6

$\frac{1}{12}$ x 36 = 3

$\frac{1}{18}$ x 36 = 2

$\frac{1}{36}$ x 36 = 1

Name ____________________ N 86 ***



Complete the table.

→	→
$\frac{1}{2}$x	+34
$\frac{1}{4}$x	+42 or x6.25
$\frac{1}{8}$x	+46 or x12.5
$\frac{1}{10}$x	+46.8
$\frac{1}{16}$x	+48 or x25
$\frac{1}{32}$x	+49 or x50
$\frac{1}{5}$x	+43.6
$\frac{1}{64}$x	+49.5 or 100x
$\frac{1}{100}$x	+49.68
-40	+58

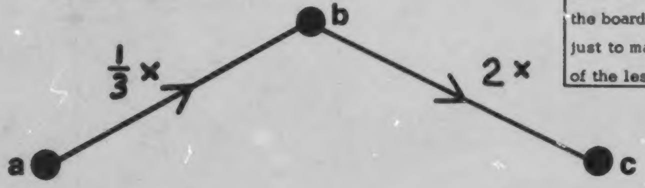
→	→
+68	$\frac{1}{2}$x
+118	$\frac{1}{3}$x
+168	$\frac{1}{4}$x
+218	$\frac{1}{5}$x
+268	$\frac{1}{6}$x
+468	$\frac{1}{10}$x
+968	$\frac{1}{20}$x
+4,968	$\frac{1}{100}$x
-82	+100
+118	-100

9.2 Fractions #2

Exercise 1

Draw this arrow picture on the board.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



Label a "9".

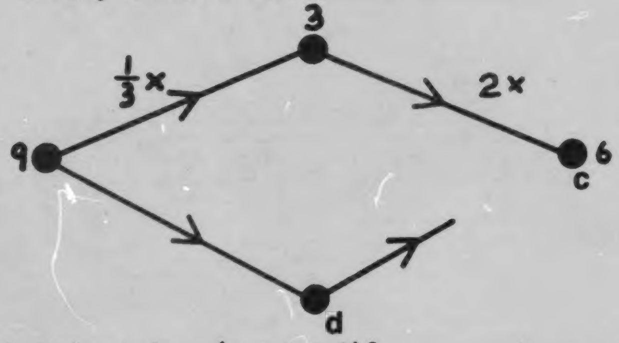
T: What number is here (point to b)?

S: 3.

T: What number is here (point to c)?

S: 6.

Extend your drawing as shown below.



T: What number is here (point to d)?

S: 18.

T: What is the ending number for this red arrow? (Point to the red arrow starting at d.)

S: 6.

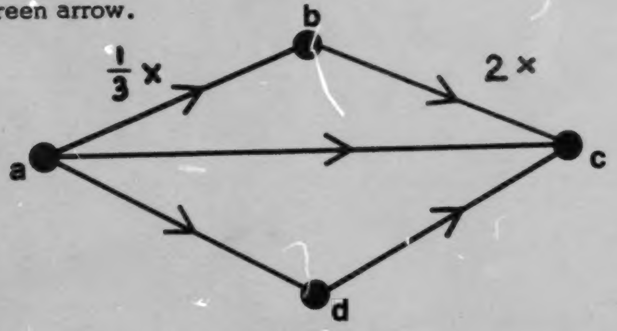
Extend the red arrow to c.

Emphasize that $\frac{1}{3}\times$ followed by $2\times$ is the same as $2\times$ followed by $\frac{1}{3}\times$ by tracing first the red-blue road and then the blue-red road, noting that in each case the ending number (6 in this example) is the same.

Continue this exercise by relabeling a "27", "45", and "30". In each case let students label the remaining dots. Answers are given in the table below.

a	b	c	d
27	9	18	54
45	15	30	90
30	10	20	60

Draw this green arrow.



T: What could this green arrow be for?

S: $\frac{2}{3}$ of (or $\frac{2}{3}\times$).

Since this way of thinking of $\frac{2}{3}\times$ may be new to your students, the suggestion to label the green arrow "$\frac{2}{3}\times$" may not be forthcoming. You should then suggest "$\frac{2}{3}\times$" yourself.

Label the green arrow "$\frac{2}{3}\times$".

T: The arrow picture shows that we can think of $\frac{2}{3}\times$ as $\frac{1}{3}\times$ followed by $2\times$ (trace the red-blue road) or as $2\times$ followed by $\frac{1}{3}\times$ (trace the blue-red road).

Write these problems on the board.

$\frac{2}{3}\times 6 = \square$ $\frac{2}{3}\times 36 = \square$ $\frac{2}{3}\times 39 = \square$

$\frac{2}{3}\times 12 = \square$ $\frac{2}{3}\times 60 = \square$ $\frac{2}{3}\times 48 = \square$

$\frac{2}{3}\times 15 = \square$ $\frac{2}{3}\times 450 = \square$

T: Copy and solve these problems. Use the arrow picture to help you.

Let the students work independently while you walk around and monitor their progress. When most of the students have finished, check the work collectively.

T: How did you calculate "$\frac{2}{3}\times 6$"?

S (pointing to the arrow picture): $\frac{1}{3}\times 6 = 2$ and $2\times 2 = 4$, so $\frac{2}{3}\times 6 = 4$.

T: Correct. Did someone do it in a different way?

S (pointing to the arrow picture): $2\times 6 = 12$ and $\frac{1}{3}\times 12 = 4$, so $\frac{2}{3}\times 6 = 4$.

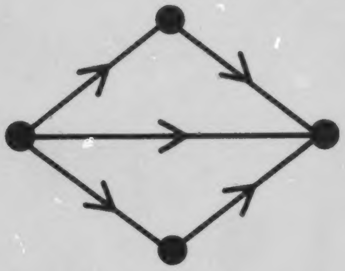
Continue in this way as long as necessary. The remaining answers are given below. As you continue to check these problems, mix the reading of "$\frac{2}{3}\times$" as "$\frac{2}{3}$ times" and "$\frac{2}{3}$ of", using the latter whenever there seems to be confusion with the "×" sign.

$\frac{2}{3}\times 12 = \boxed{8}$ $\frac{2}{3}\times 36 = \boxed{24}$ $\frac{2}{3}\times 39 = \boxed{26}$

$\frac{2}{3}\times 15 = \boxed{10}$ $\frac{2}{3}\times 60 = \boxed{40}$ $\frac{2}{3}\times 48 = \boxed{32}$

$\frac{2}{3}\times 450 = \boxed{300}$

Erase all numerals and arrow labels from the arrow picture.



T: I want to use this diagram to help me calculate "$\frac{3}{4}\times 100$". What should the red arrow be labeled?

S: $\frac{1}{4}\times$ (or $3\times$).

Write "$\frac{1}{4}\times$" in red on the board.

T: What should the blue arrow be labeled?

S: $3\times$ (or $\frac{1}{4}\times$).

Write "$3\times$" in blue on the board.

T: What could the green arrow be for?

S: $\frac{3}{4}\times$.

Write "$\frac{3}{4}\times$" in green on the board.

T: Now, let's use this arrow picture to solve my problem.

Write "$\frac{3}{4} \times 100 = \square$" on the board.

T: What is $\frac{1}{4} \times 100$?

S: 25.

T: And what is 3×25?

S: 75.

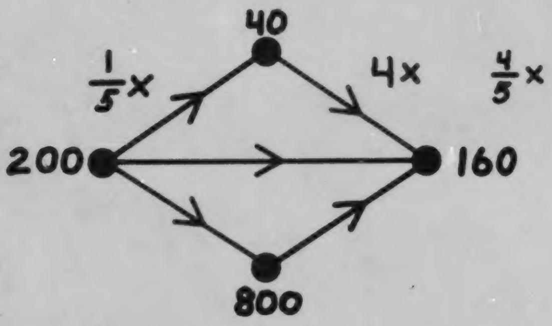
T: So what is $\frac{3}{4} \times 100$?

S: 75.

T: What is $\frac{3}{4} \times 24$?

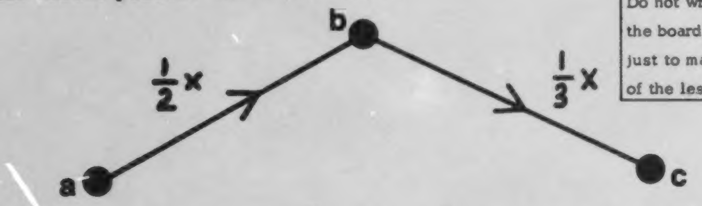
S: $\frac{1}{4} \times 24 = 6$ and $3 \times 6 = 18$, so $\frac{3}{4} \times 24 = 18$.

Erase the arrow labels and repeat the exercise for "$\frac{4}{5} \times 200 = \square$". An arrow picture for this calculation is shown below.



Exercise 2

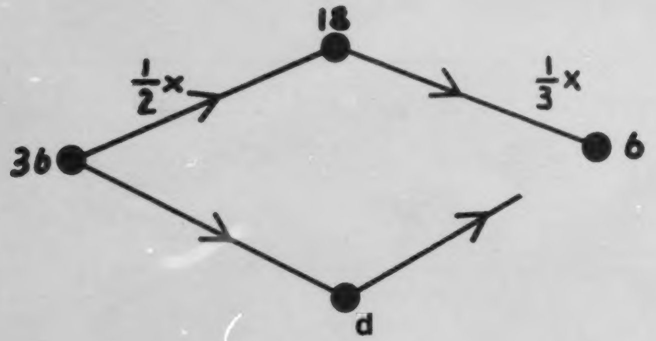
Draw this arrow picture on the board.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: In this picture the red arrow is for $\frac{1}{2}$ × and the blue arrow is for $\frac{1}{3}$ ×.

Label a "36" and let students label b and c. Then extend your drawing as below.



T: What number is here (point to d)?

S: 12.

T: What is the ending number for this red arrow? (Point to the arrow starting at d.)

S: 6, because $\frac{1}{2} \times 12 = 6$.

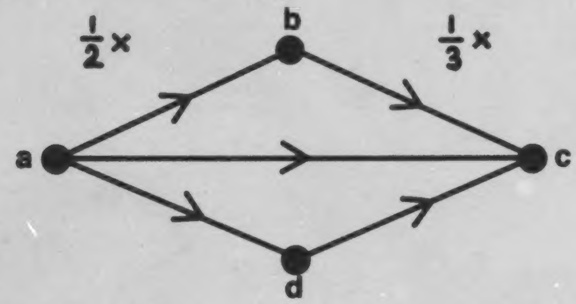
Extend the red arrow to c.

Emphasize that the ending number is the same whether one takes the red ($\frac{1}{2}$ ×) followed by blue ($\frac{1}{3}$ ×) road or the blue ($\frac{1}{3}$ ×) followed by red ($\frac{1}{2}$ ×) road.

Repeat the exercise using 72 and 30. Solutions are provided in the table below.

a	b	c	d
36	18	6	12
72	36	12	24
30	15	5	10

Add this green arrow to your drawing.



Trace the arrow from c to b.

T: What could this return arrow be for?

S: $\div\frac{1}{3}$.

T: That's correct, but there is another name, too. What is another name for this return arrow?

S: "3×".

Trace the arrow from b to a.

T: What could this return arrow be for?

S: 2×.

Trace the arrow from c to a.

T: What could this return arrow be for?

S: 6x.

T: Good. So what could the green arrow (trace the arrow from a to c) be for?

S: $\frac{1}{6}$x.

Write "$\frac{1}{6}$x" in green on the board. Also write these problems on the board.

$$\frac{1}{6} \times 42 = \square \qquad \frac{1}{6} \times 75 = \square$$

$$\frac{1}{6} \times 48 = \square \qquad \frac{1}{6} \times 45 = \square$$

T: Copy and solve these problems. You may use the arrow picture to help you.

After a few minutes of individual work, solve these problems collectively.

T: What is $\frac{1}{6} \times 42$? (7) How did you get 7?

S: $\frac{1}{2} \times 42 = 21$ and $\frac{1}{3} \times 21 = 7$, so $\frac{1}{6} \times 42 = 7$.[*]

T: What is $\frac{1}{6} \times 48$?

S: I used the red arrow first. $\frac{1}{2} \times 48 = 24$ and $\frac{1}{3} \times 24 = 8$, so $\frac{1}{6} \times 48 = 8$.

---

[*] A student might suggest that $\frac{1}{6} \times 42 = 7$ because $6 \times 7 = 42$. This would be a good explanation and should be accepted. However, you will also want to suggest using the $\frac{1}{2}$x followed by $\frac{1}{3}$x arrows.

T: Very good. What is $\frac{1}{6} \times 75$?

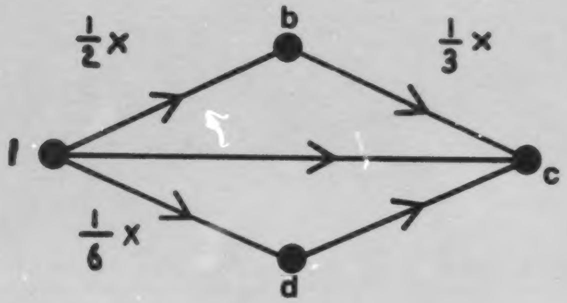
S: I used the blue arrow first. $\frac{1}{3} \times 75 = 25$ and $\frac{1}{2} \times 25 = 12.5$, so $\frac{1}{6} \times 75 = 12.5$.

T: Good. What is $\frac{1}{6} \times 45$?

S: I used the blue arrow first again. $\frac{1}{3} \times 45 = 15$ and $\frac{1}{2} \times 15 = 7.5$, so $\frac{1}{6} \times 45 = 7.5$.

Erase any numerals you may have in your picture but leave the arrow labels.

Label a "1".



T: Suppose we label the left-hand dot "1". What number is here (point to b)?

S: $\frac{1}{2}$ or 0.5.

Label b "$\frac{1}{2}$".

T: What number is here (point to d)?

S: $\frac{1}{3}$.

Label d "$\frac{1}{3}$".

T: What number is here (point to c)?

S: $\frac{1}{6}$.

Label c "$\frac{1}{6}$".

T (tracing the arrow from b to c): What calculation does this blue arrow show?

S: "$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$".

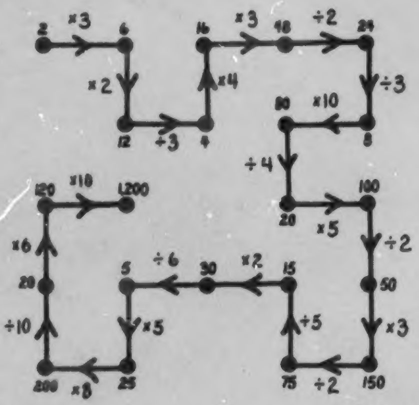
T (tracing the arrow from d to c): What calculation does this red arrow show?

S: "$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$".

Worksheets N87, N88, and N89 are available.

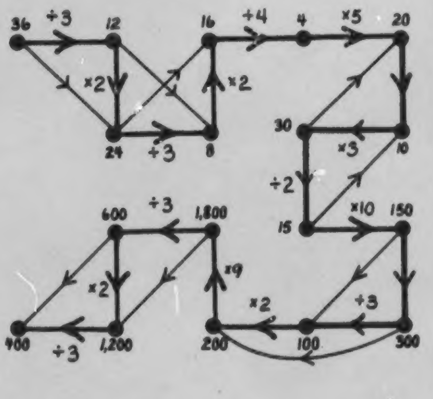
Name ____________ **N 87** *

Label each arrow "x ☐" or "÷ ☐".


2
×3
6
×2
12
÷3
4
×4
16
×3
48
÷2
24
÷3
8
×10
80
÷4
20
×5
100
÷2
50
×3
150
÷2
75
÷5
15
×2
30
÷6
5
×5
25
×8
200
÷10
20
×6
120
×10
1,200

Name ____________ **N 88** **

Label each arrow "x ☐" or "÷ ☐".

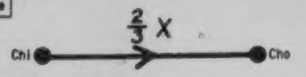

36
÷3
12
×2
24
÷3
8
16
×2
÷4
4
×5
20
30
×3
10
÷2
15
×10
150
600
÷3
1,800
×2
×9
400
÷3
1,200
200
×2
100
÷3
300

Use blue to draw the nine $\frac{2}{3}$x arrows in this picture.

Name ____________ **N 89** ***

Cho is a secret WHOLE number.

First clue


$\frac{2}{3}$ x
Chi
Cho

Chi and Cho could be: (Some are done for you.)
Many other solutions are possible for the first clue.

Chi	Cho	Chi	Cho	Chi	Cho	Chi	Cho
3	2	12	8	10.5	7	27	18
6	4	15	10	18	12	30	20
9	6	24	16	21	14	33	22

Second clue

Chi is a multiple of 9 between 50 and 100.

Chi and Cho could be:

Chi	Cho	Chi	Cho	Chi	Cho
54	36	72	48	90	60
63	42	81	54	99	66

Third clue

Cho is a square number.

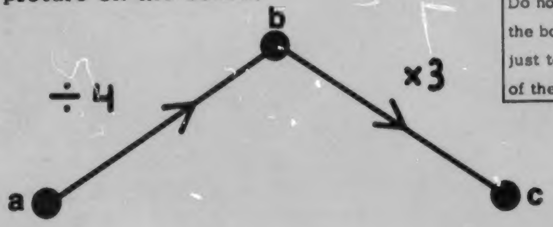
Chi is 54.   Cho is 36.

9.3 Fractions #3

Exercise 1

Draw this arrow picture on the board.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.



Label a "16".

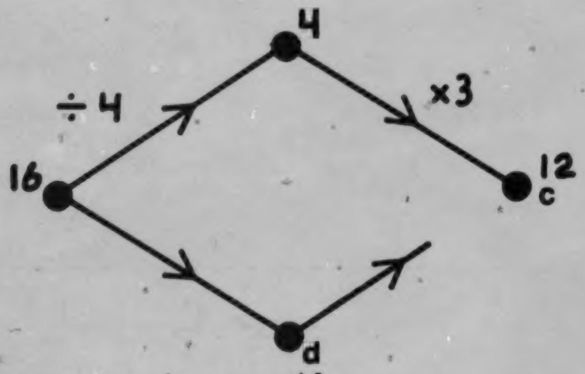
T: What number is here (point to b)?

S: 4.

T: What number is here (point to c)?

S: 12.

Expand your drawing as shown below.



T: What number is here (point to d)?

S: 48.

T: What is the ending number for this red arrow? (Point to the red arrow starting at d.)

S: 12.

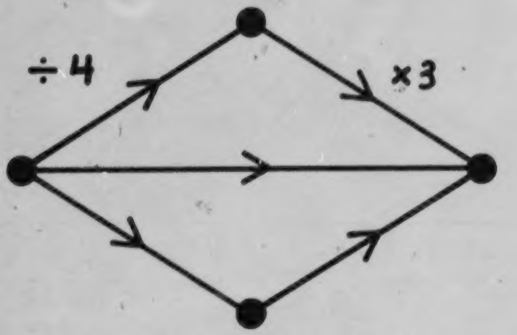
Extend the red arrow to c.

Emphasize that ÷ 4 followed by × 3 is the same as × 3 followed by ÷ 4 by tracing first the red-blue road and then the blue-red road, noting that in each case the number at c (12 in this example) is the same.

Continue this activity by relabeling a "100", "36", and "12". In each example have students label the other dots. Answers are given in the table below.

a	b	c	d
100	25	75	300
36	9	27	108
12	3	9	36

Draw this green arrow.



T: What could this green arrow be for?

S: $\frac{3}{4}$ ×.

Since this way of thinking of $\frac{3}{4}\times$ may still be new to many students, the suggestion to label the green arrow "$\frac{3}{4}\times$" may not be forthcoming. You should then suggest "$\frac{3}{4}\times$" yourself. (Read "$\frac{3}{4}\times$" as "$\frac{3}{4}$ of" or "$\frac{3}{4}$ times".)

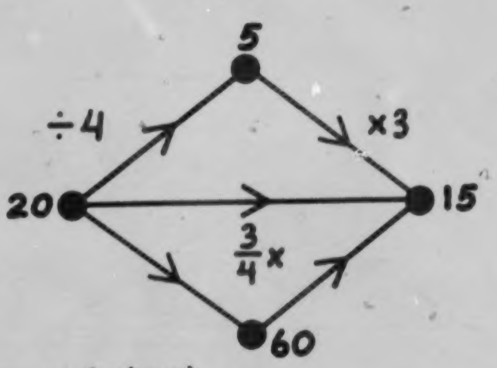
Label the green arrow "$\frac{3}{4}\times$".

T: The arrow picture shows that finding $\frac{3}{4}$ of a number is the same as dividing the number by four and multiplying the result by three (trace the red-blue road), or multiplying the number by three and dividing the result by four (trace the blue-red road).

Write this problem on the board and ask the students to solve it on their paper.

$$\frac{3}{4} \times 20 = \square$$

While the students are working on this problem, ask a student who is having difficulty to come to the board, place 20 in the arrow picture, and label the remaining dots.



Write these problems on the board.

$$\frac{3}{4} \times 40 = \square \qquad \frac{3}{4} \times 200 = \square$$

$$\frac{3}{4} \times 36 = \square \qquad \frac{3}{4} \times 60 = \square$$

T: Copy and solve these problems. Use the arrow picture to help you.

Let the students work independently. When most of the students have finished, check the work collectively.

T: How did you calculate "$\frac{3}{4} \times 40$"?

S (pointing to the arrow picture): $40 \div 4 = 10$ and $10 \times 3 = 30$, so $\frac{3}{4} \times 40 = 30$.

S: I went the other way. $40 \times 3 = 120$ and $120 \div 4 = 30$, so $\frac{3}{4} \times 40 = 30$.

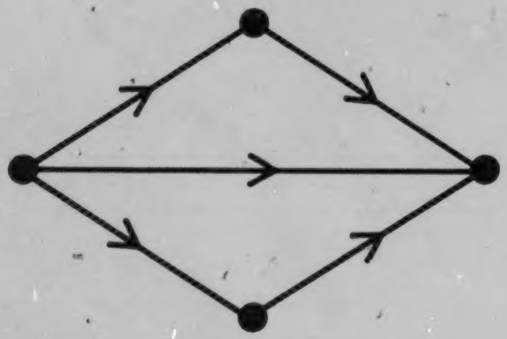
Continue as long as necessary. The remaining answers are given below.

$$\frac{3}{4} \times 36 = \boxed{27}$$

$$\frac{3}{4} \times 200 = \boxed{150}$$

$$\frac{3}{4} \times 60 = \boxed{45}$$

Erase all numerals and arrow labels from your picture.



T: How could we use this diagram to help calculate "$\frac{2}{3} \times 12$"? What could the arrows be for?

Write "$\frac{2}{3} \times 12 = \square$" on the board.

S: The red arrows could be for ÷ 3.

S: The blue arrows could be for ×2.

S: Then the green arrow would be for $\frac{2}{3}\times$.

T: What is $\frac{2}{3} \times 12$?

S: $12 \div 3 = 4$ and $4 \times 2 = 8$, so $\frac{2}{3} \times 12 = 8$.

Write these problems on the board.

$\frac{3}{7} \times 28 = \square$ $\frac{5}{6} \times 18 = \square$

$\frac{2}{5} \times 10 = \square$ $\frac{5}{8} \times 16 = \square$

T: Copy and solve these problems. Use an arrow picture to help you.

After a few minutes, ask:

T: How did you calculate "$\frac{3}{7} \times 28$"?

S: I divided by 7 and multiplied the result by 3.

S: $28 \div 7 = 4$ and $4 \times 3 = 12$, so $\frac{3}{7} \times 28 = 12$.

Continue checking these problems collectively.

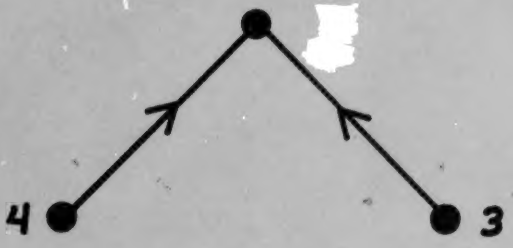
$$\frac{2}{5} \times 10 = \boxed{4}$$

$$\frac{5}{6} \times 18 = \boxed{15}$$

$$\frac{5}{8} \times 16 = \boxed{10}$$

Exercise 2

Draw this arrow picture on the board.



T: What could the red and blue arrows be for?

S: The red arrow could be for +1 and the blue arrow could be for +2.

T: Why?

S: 4 + 1 = 5 and 3 + 2 = 5.

T: Good. What else could the arrows be for?

S: Red: +10 and blue: +11.

S: Red: +2 and blue: × 2.

S: Red: 3 × and blue: 4 ×.

The above responses are just a few that were given in our development classes. There are many possible answers. Accept any that are correct.

T: Now label both arrows using "×☐" (Read: "times some number").
I will draw a chart on the board to record the answers.

Draw this chart on the board.

→	→
3×	4×

T: What else could the arrows be for?

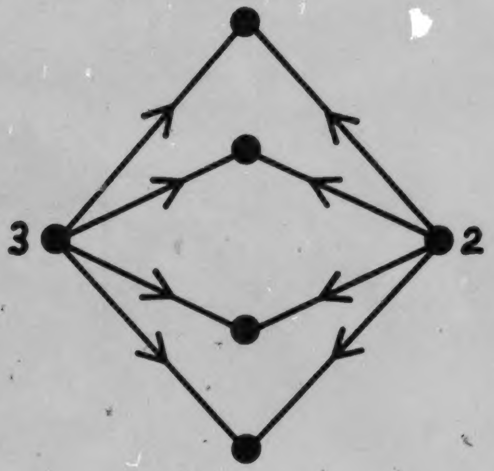
As students respond, record their responses in the chart. A partially complete chart is shown below.

→	→
3×	4×
6×	8×
9×	12×
12×	16×
15×	20×
18×	24×
21×	28×

It may happen that your chart will not be as orderly and suggestive as the one above. Don't worry; just accept any reasonable comments made about the chart.

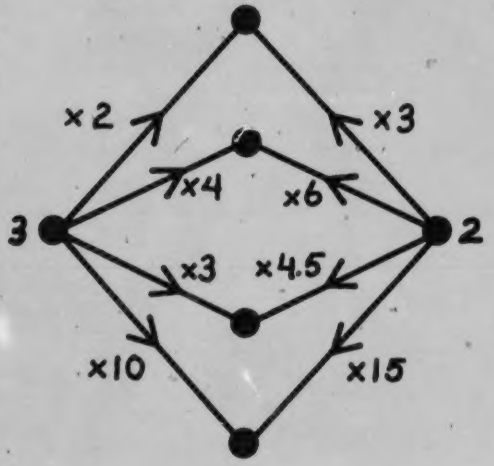
Exercise 3

Draw this arrow picture on the board.

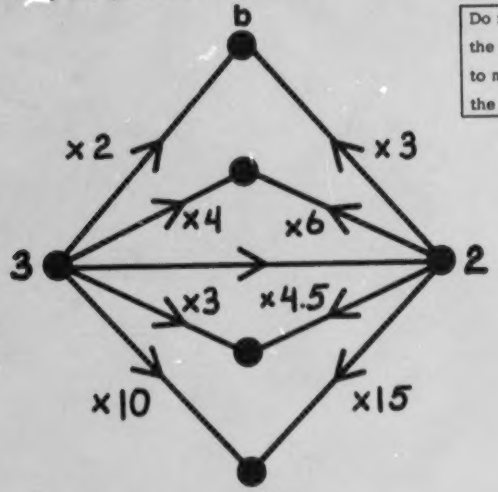


T: Label each red arrow "×☐". Each arrow may be a different relation.

Have students label the arrows. There are many possibilities. Suppose your picture looks like the one below.



Add this blue arrow to your picture.



Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

T: What could the blue arrow be for?

Since this is yet another way to think of $\frac{2}{3}\times$, it may be necessary to suggest:

Start at 3 and trace the × 2 arrow. Then trace a return arrow from b to 2.

T: What could this return arrow be for?

S: $\div 3$ (or $\frac{1}{3}\times$).

T: What could the blue arrow be for?

S: $\frac{2}{3}\times$.

T: Are there other names for the blue arrow?

S (pointing to the × 4 arrow followed by a ÷ 6 return arrow): $\frac{4}{6}\times$.

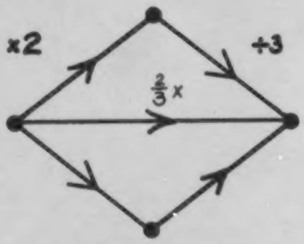
S (pointing to the × 10 arrow followed by a ÷ 15 return arrow): $\frac{10}{15}\times$.

S (pointing to the × 3 arrow followed by a ÷ 4.5 arrow): $\frac{3}{4.5}$ x.

Something like the last example, $\frac{3}{4.5}$ x, may or may not occur in your class. If it does, accept it. If it does not occur, there is no need to mention it at this time.

Worksheets N90, N91, N92, and N93 are available.

Name ______________ **N 90 ***



Label the blue arrow and solve these problems.

$\frac{2}{3}$ x 6 = 4

$\frac{2}{3}$ x 9 = 6

$\frac{2}{3}$ x 15 = 10

$\frac{2}{3}$ x 30 = 20

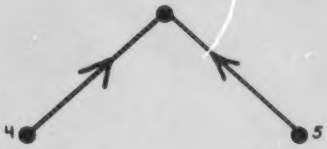
$\frac{2}{3}$ x 60 = 40

$\frac{2}{3}$ x 12 = 8

$\frac{2}{3}$ x 24 = 16

$\frac{2}{3}$ x 45 = 30

Name ______________ **N 91 ****



Complete these charts by labeling the red arrow and the blue arrow "x☐".

→	→
×5	×4
×10	×8
×20	×16
×15	×12
×25	×20

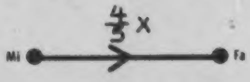
→	→
×30	×24
×35	×28
×40	×32
×45	×36
×50	×40

Many other solutions are possible.

Name ______________ **N 92 *****

Mi and Fa are two secret WHOLE numbers.

First clue



Mi and Fa could be:
(Many other solutions are possible for this clue.)

Mi	Fa	Mi	Fa	Mi	Fa
5	4	20	16	35	28
10	8	25	20	40	32
15	12	30	24	and so on.	

Mi + Fa could be 9 ; 18 ; 27 ; 36 ; 45 ; 54 ; 63 ; 72 ; and so on. (Many other solutions are possible for this clue.)

What do you notice about Mi + Fa ? Write your answer below.

Mi + Fa is a multiple of nine.

Second clue

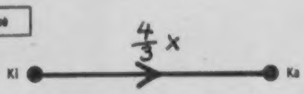
Mi + Fa = 81.

Mi is 45 . Fa is 36 .

Name ______________ **N 93 ******

Ki and Ka are two secret WHOLE numbers.

First clue



Ki and Ka could be: (Many other solutions are possible for this clue.)

Ki	Ka	Ki	Ka	Ki	Ka
18	24	36	48	66	88
27	36	75	100	99	132
30	40	72	96	102	136
				and so on.	

Ki + Ka could be 42 ; 63 ; 70 ; 77 ; 84 ; 175 ; 168 ; 154 ; 231 ; 238 245 ; 252 ; and so on. (Many other solutions are possible for this clue.)

What do you notice about Ki + Ka ? Write your answer below.

Ki + Ka is a multiple of seven.

Second clue

Ki + Ka is between 100 and 110.

Ki + Ka = 105 . Ki is 45 . Ka is 60 .

10

BINARY ABACUS

This one lesson chapter extends an idea that first appeared at the end of the "Clinton Street" story-workbook. The binary system (base two positional system) of numeration is explored via a binary abacus. No attempt to write base two names for numbers is made except on the binary abacus. This way of dealing with non-decimal systems of numeration will be continued in Part III of the CSMP Mathematics for Intermediate Grades Program.

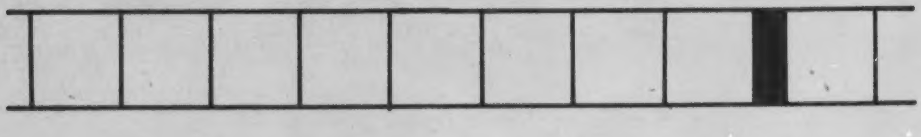
CAPSULE LESSON SUMMARIES

Section 10.1: Review the simple rule of the binary abacus, and read numbers on the binary abacus making use of this rule. Determine values for the squares of the binary abacus both left and right of the bar, and represent numbers on the binary abacus. Examine the relationship between binary representation of a number and the subsets of a given set. (Worksheets N94, N95, N96)

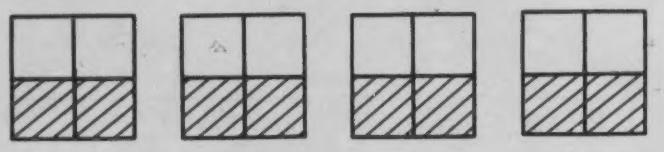
10.1 Binary Abacus

Exercise 1

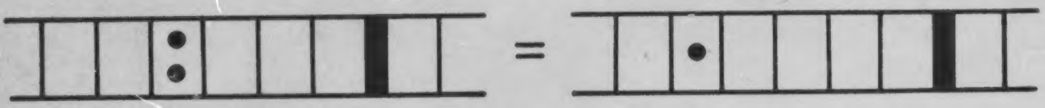
Draw this part of a binary abacus on the board.



NOTE: If your chalkboard is not magnetic, a magnetic abacus can be constructed using your demonstration Minicomputer boards. Cover the red and white squares with paper and use only the brown and purple squares on each board.

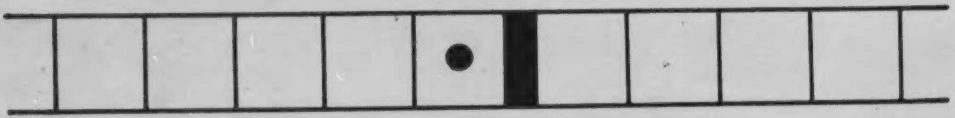


T: Do you remember the binary snake or binary abacus in "Clinton Street"? For this binary abacus there is only one rule:



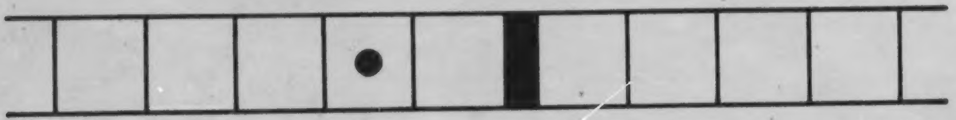
Two checkers on a square represents the same number as one checker on the next square to the left.

Put a checker on the abacus.



T: This is the number 1.

Move the checker one square to the left.

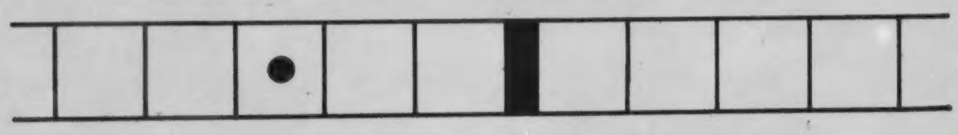


T: What number is this?

S: 2.

T: Suppose I move the checker again?

Show the move on your boards.



S: 4.

T: And again?

S: 8.

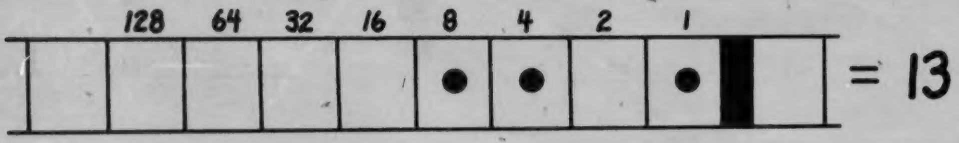
T: Again?

S: 16.

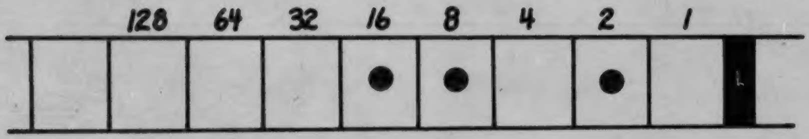
Continue with 32; 64; 128; 256; 512; 1,024; ... Indicate to the students that the binary abacus goes on.

T: Who can put 13 on the binary abacus using at most one checker on each square?

Have a student put on 13.



T: Good. Now I will move each checker one square to the left.



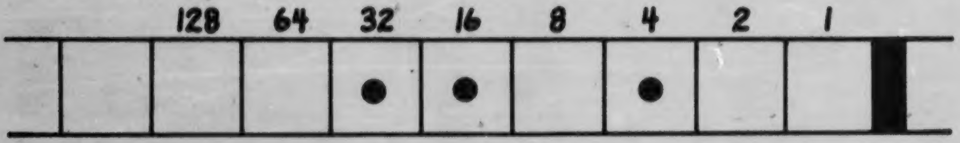
T: What number is on the abacus now?

S: 26.

T: Why?

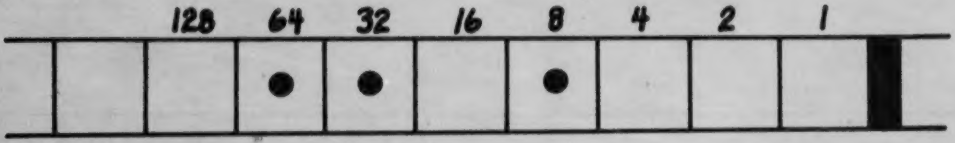
S: By moving all the checkers one square to the left, you double the number.

T: Suppose I move them one square to the left again?


128
64
32
16
8
4
2
1

S: 52.

T: Again?


128
64
32
16
8
4
2
1

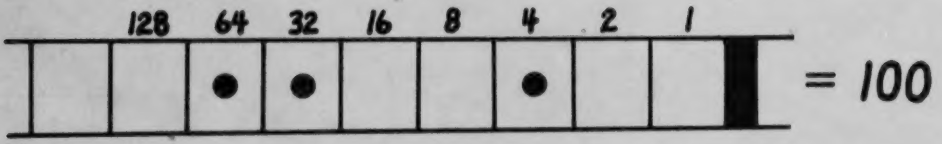
S: 104.

T: Check it to be sure.

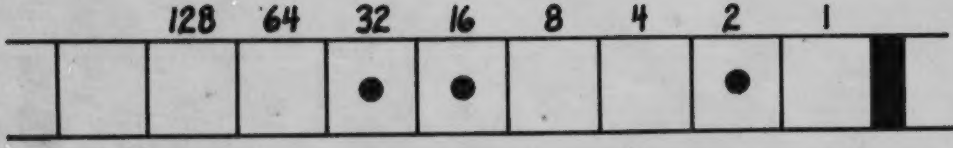
S: 64 + 32 + 8 = 104.

Clear the abacus.

T: Who can put on 100?


128
64
32
16
8
4
2
1
= 100

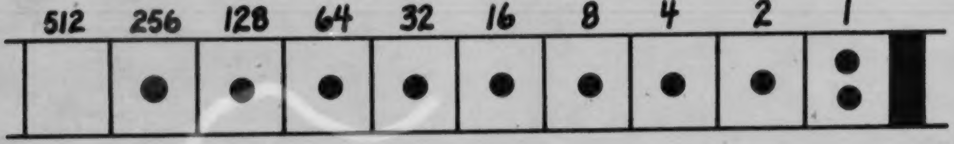
T: I move each checker one square to the right. What number is on the abacus now?



S: 50, because moving the checkers one square to the right is the same as taking half of the number.

Exercise 2

Display this configuration on the abacus.



T: What number is on the abacus?

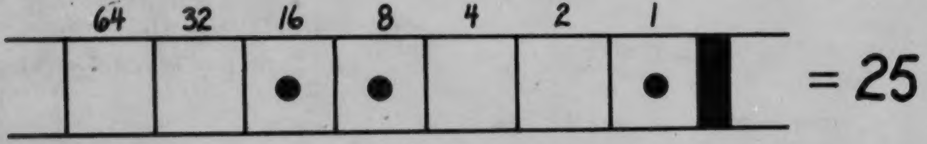
S: 512.

T: How did you get 512?

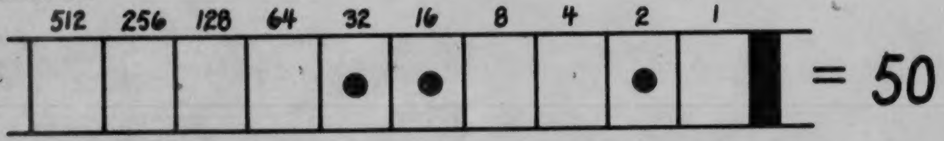
S: Exchange the two checkers on the 1 - square for one checker on the 2 - square. But then we'd have two checkers on the 2 - square which is the same as one checker on the 4 - square. But then we'd have two checkers on the 4 - square which is the same as one on the 8 - square. Keep doing this until we get two checkers on the 256 - square which is the same as one checker on the 512 - square.

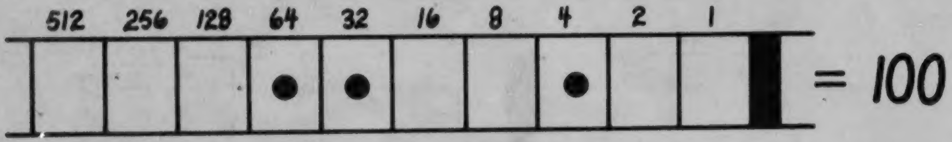
T: On a sheet of paper draw a binary abacus and put on 25. Remember, use at most one checker on each square.

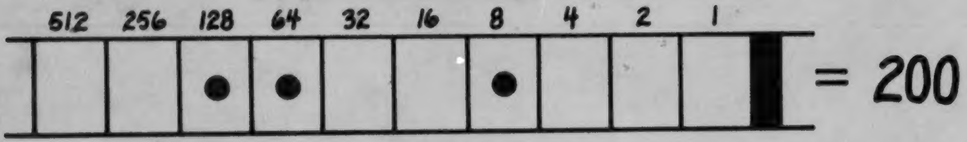
Have a student put 25 on the abacus.


64
32
16
8
4
2
1
= 25

Continue the exercise by asking students to write 50; 100; and 200 on the abacus.

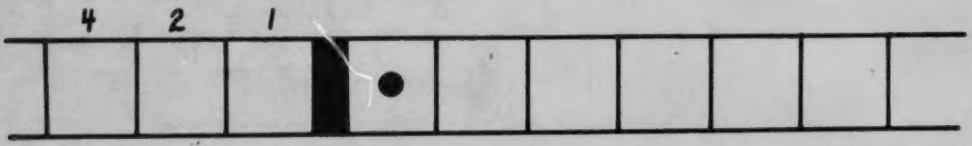

512
256
128
64
32
16
8
4
2
1
= 50


512
256
128
64
32
16
8
4
2
1
= 100


512
256
128
64
32
16
8
4
2
1
= 200

Exercise 3

Put this checker on the abacus and remind the students that the checker is to the right of the bar.

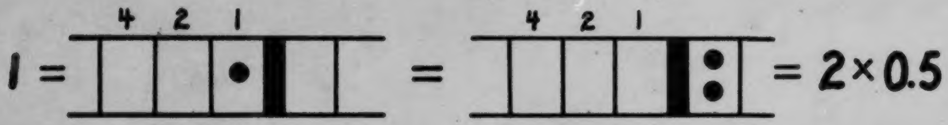


T: Write this number on your paper.

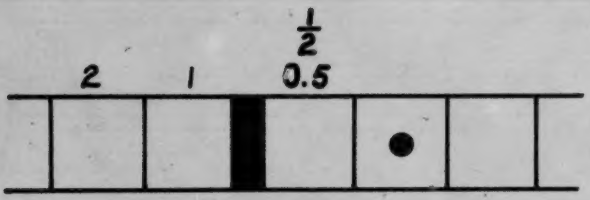
S: 0.5.

S: $\frac{1}{2}$.

If some students suggest 0.8, 0.1, or 0.2, remind them that we are working with a binary abacus and not the Minicomputer. Show:


1 =
4
2
1
=
4
2
1
= 2×0.5

Label the square and move the checker one square to the right.

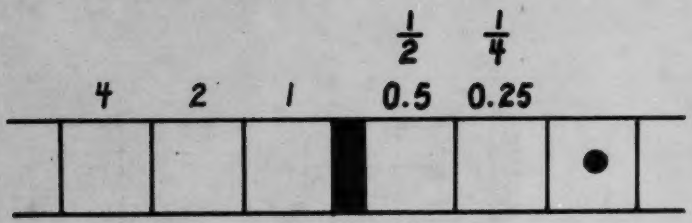

2
1
$\frac{1}{2}$
0.5

T: What number is this?

S: 0.25.

S: $\frac{1}{4}$.

Again, move the checker one square to the right.


4
2
1
$\frac{1}{2}$
0.5
$\frac{1}{4}$
0.25

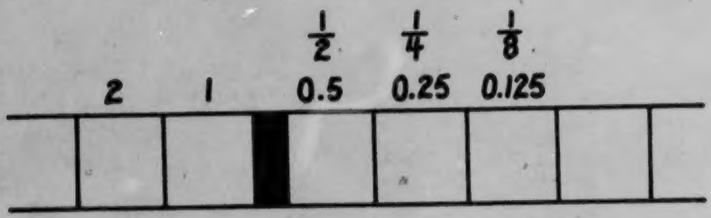
T: What is this number?

S: One-half of 0.25.

S: 0.25 = 0.250 and one-half of 0.250 is 0.125. It is 0.125.

S: $\frac{1}{8}$.

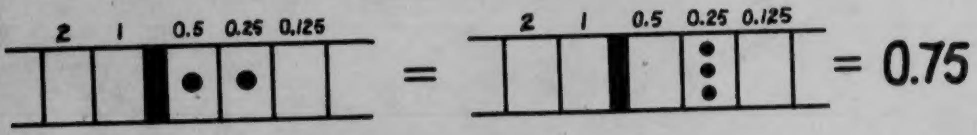
Add these labels to your drawing.



1/2
1/4
1/8
2
1
0.5
0.25
0.125

T: Draw an abacus on your paper and put on 0.75.

After a minute or two let a student put on 0.75. There are two likely answers.

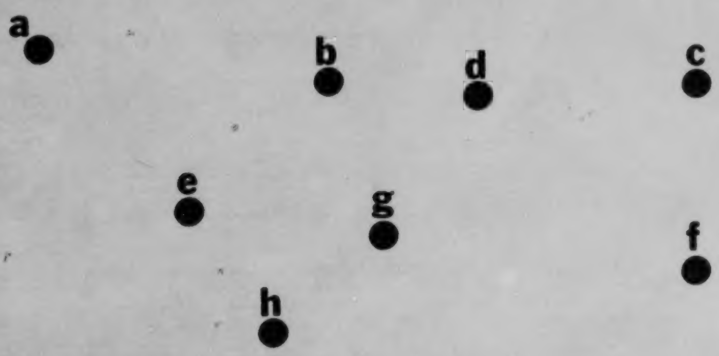


2
1
0.5
0.25
0.125
=
2
1
0.5
0.25
0.125
= 0.75

Accept any correct answer. The first configuration above is standard because it uses at most one checker on each square.

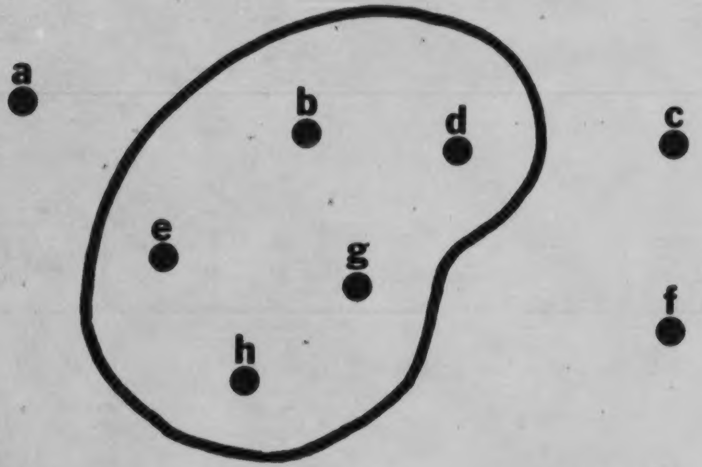
Exercise 4

Draw and label these dots on the chalkboard.



T: I want to tell you a story about Nabu. Nabu has eight friends. I have shown each friend with a dot and a letter. (Point to the drawing on the board.) Each Sunday Nabu is allowed to invite some of his friends to his house. We will show the friends he invites in a string. Show me a set of friends that Nabu could invite next Sunday.

Have a student come to the board and draw a string around any set of friends. Suppose the student makes the following choice.

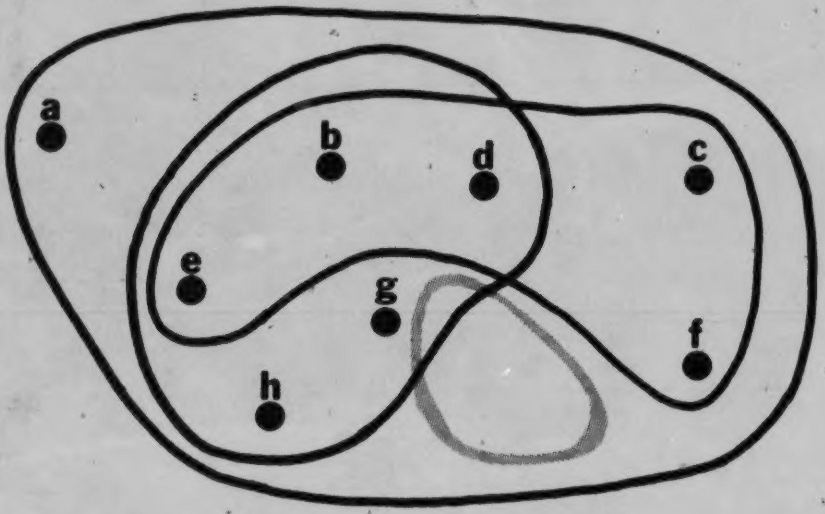


T: Good. The red string indicates that next Sunday Nabu will invite friends b, d, e, g, and h. On the following Sunday, Nabu again will be permitted to invite some of his friends, but he must follow this rule:

> Each Sunday Nabu must invite a different set or combination of friends.

Show us some sets of friends that Nabu could invite.

Let students come to the board and draw different color strings to represent different sets of friends. There are many possibilities, but suppose your picture looks like the one below.



NOTE: The grey string may not be suggested. It is correct and indicates that Nabu invited no friends. If it is not suggested here, you may have to suggest it yourself later in the lesson.

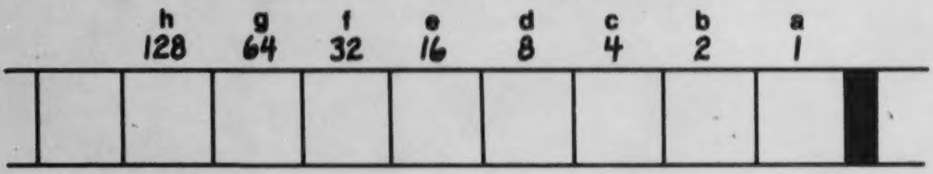
T: Look at the red and black strings. On some Sunday Nabu could invite the friends in the red string and on another Sunday he could invite the friends in the black string. He may invite the same friends, but never the same set or combination of friends.

T: How many Sundays will it take to invite all the possible combinations of friends?

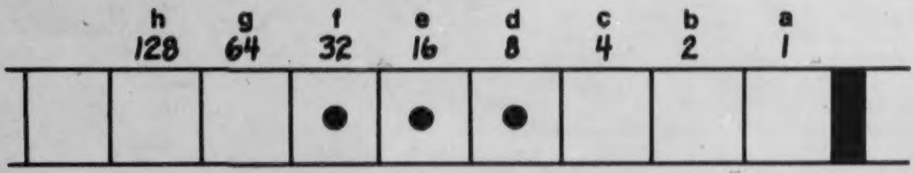
Accept any estimate that is volunteered. (You might want to list some on the board for the sake of comparison later.)

T: To solve this problem Nabu used the binary abacus.

Label the boards of the abacus as shown below.


h 128
g 64
f 32
e 16
d 8
c 4
b 2
a 1

T: Friends d, e, and f came over one Sunday. Nabu put three checkers on the abacus.

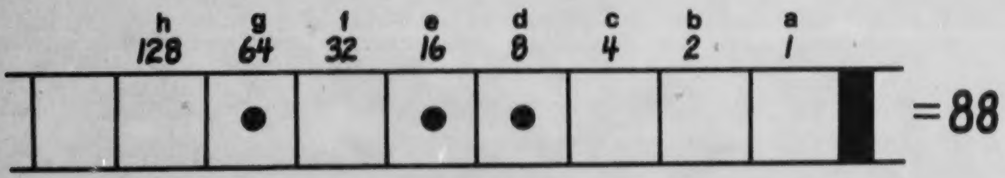

h 128
g 64
f 32
e 16
d 8
c 4
b 2
a 1

T: He also wrote "56" in his notebook. Why did he write "56"?

S: That's the number on the abacus. 8 + 16 + 32 = 56.

T: He wrote "88" in his notebook another Sunday. Do you know which friends he invited?

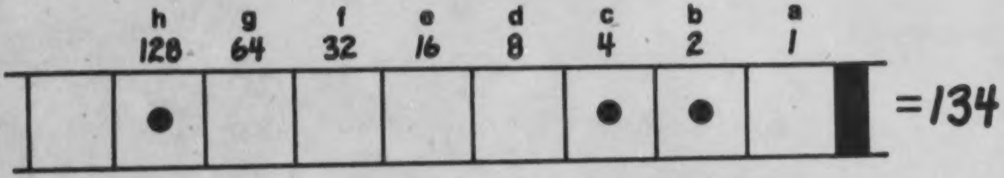
Have a student show the number 88 on the abacus.


h 128
g 64
f 32
e 16
d 8
c 4
b 2
a 1
=88

S: He invited g, e and d.

T: On still another Sunday he invited h, c, and b. What number did he write in his notebook that day?

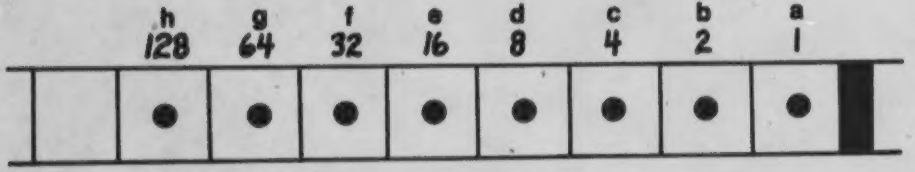
Have a student show the number on the abacus.


h
128
g
64
f
32
e
16
d
8
c
4
b
2
a
1
=134

S: The number is 134, because 128 + 4 + 2 = 134.

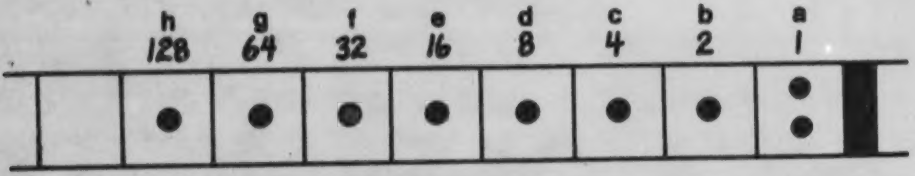
T: What is the largest number he could write?

S: The one for all his friends -- one checker on each square.


h
128
g
64
f
32
e
16
d
8
c
4
b
2
a
1

S: 255.

This answer can be computed quite easily by addition. Another method, based on Exercise 2, may also be used. Add one checker to the 1 - square.


h
128
g
64
f
32
e
16
d
8
c
4
b
2
a
1

T: Using what we discovered earlier, what number is this?

S: 256.

T: But there is an extra checker on this square (point to 1 - square). So what is the largest number Nabu could write?

S: 255.

T: Good. The largest number is 255. What is the smallest number Nabu could write?

S: 0, when he invited no friends.

T: How many numbers could he write down?

S: 256. (0 through 255.)

T: And how many Sundays?

S: The same, 256.

T: About how many years would it take Nabu to invite all the possible combinations of his friends?

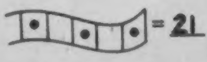
S: About 5 years, because there are 52 Sundays per year.

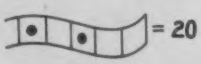
Worksheets N94, N95, and N96 are available.

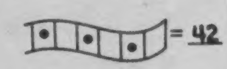
Name

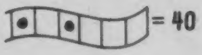
N 94 *

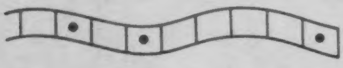
ON THE BINARY SNAKE

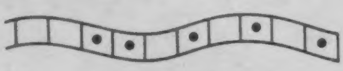

= 21

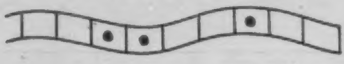

= 20


= 42


= 40


= 161


= 213


= 100

Name

N 95 **

Last Sunday, Nabu wrote "30" in his notebook.
Which friends did he invite? b, c, d, e

		h	g	f	e	d	c	b	a
					●	●	●	●	

On another day he wrote "75" in his notebook.
Which friends did he invite? a, b, d, g

		h	g	f	e	d	c	b	a
			●			●		●	●

The day Nabu wrote "150" in his notebook, which friends did he invite?
b, c, e, h

		h	g	f	e	d	c	b	a
		●			●		●	●	

The day Nabu wrote "200" in his notebook, which friends did he invite?
d, g, h

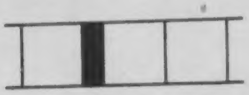
		h	g	f	e	d	c	b	a
		●	●			●			

Name

N 96 ***

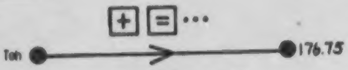
First clue

Toh can be put on this part of the binary abacus using at most one checker on each board.



Toh is 1 or 1.5 or 1.75 or 0.5 or 0.75 or 0.25 or 1.25 or 0.

Second clue



Toh is 0.25 or 1.75.

Third clue

If you multiply Toh by 28, you will find a square number.

Toh is 1.75.

Comprehensive School Mathematics Program

CEMREL, Inc.

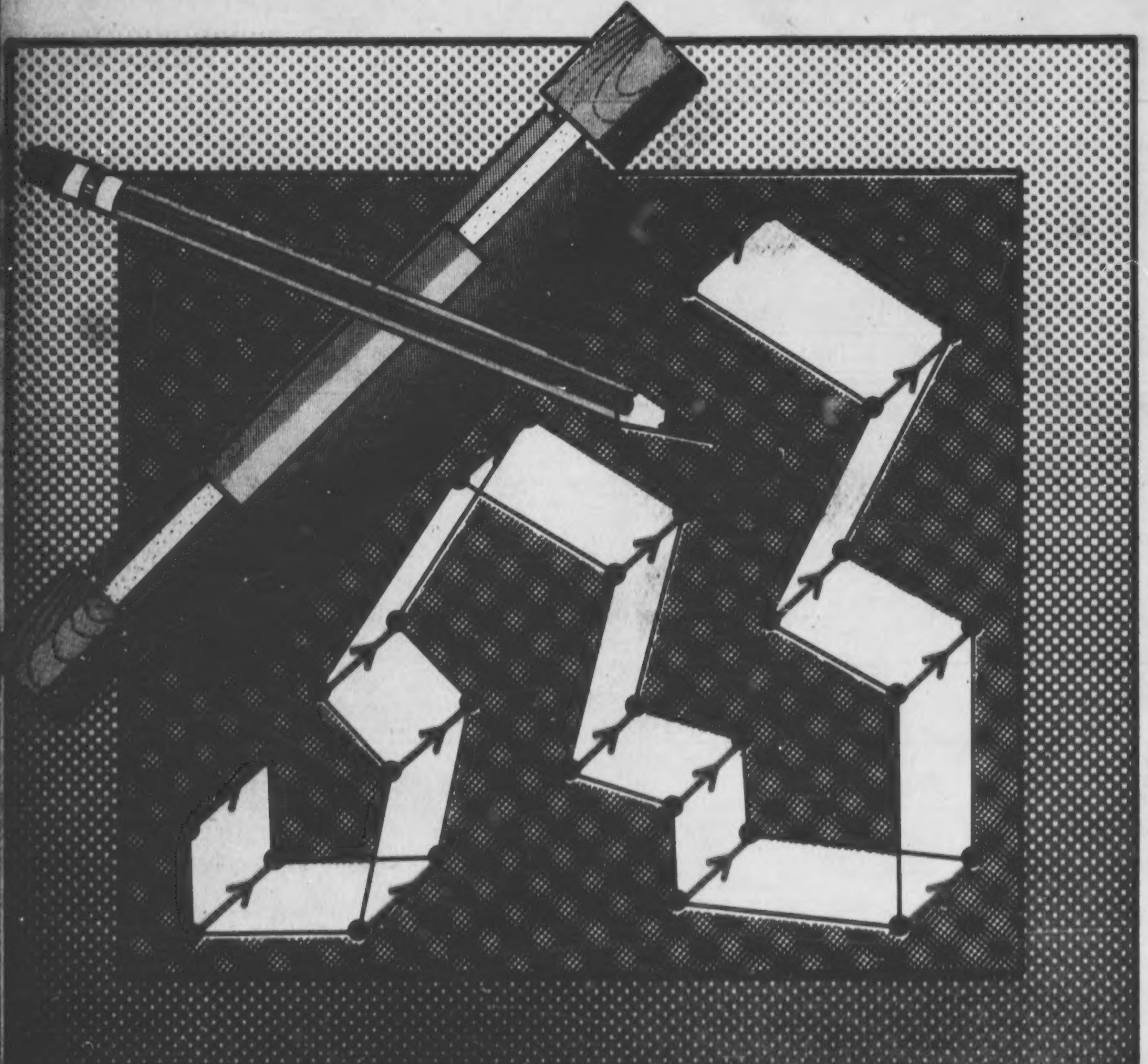
4-23001

CSMP Mathematics for the Intermediate Grades

Part II

Name

Worksheets



WHAT'S IN THIS BOOK ?

This book contains all the worksheets you will need this semester. They are in the following order:

1. All the "N" worksheets: N01 through N107
2. All the "L" worksheets: L01 through L22
3. All the "G" worksheets: G01 through G22
4. All the "P" worksheets: P01 through P04

HOW DO YOU REMOVE PAGES ?

Step 1: Flip through until you find the page you want. The number of each worksheet is in the top right corner of the page.

Step 2: Fold and crease the page along the perforation line.

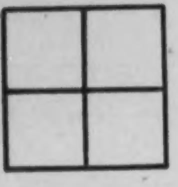
Step 3: With your left hand index finger and thumb, press book firmly down left of the perforation line and along center fold, starting at the top of the book.

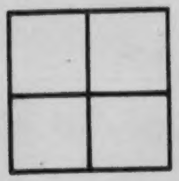
Step 4: With your right hand holding top right corner of page, tear page along the perforation line. Follow the tear with your left hand holding down center fold of book.

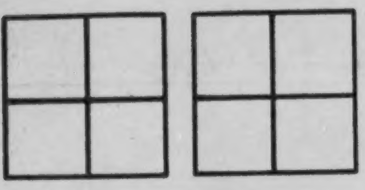
Name ______________________________

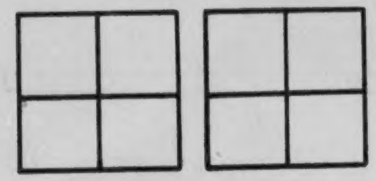
N 01 *

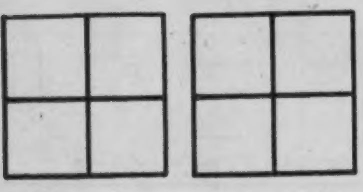
Put any number you wish on the Minicomputer using exactly four positive checkers.

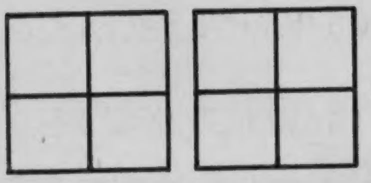

= ______

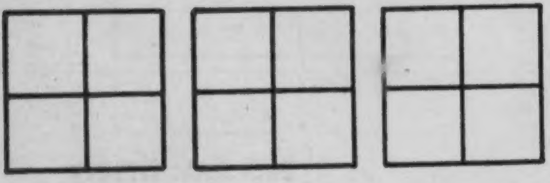

= ______

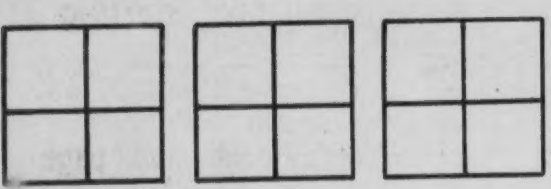

= ______

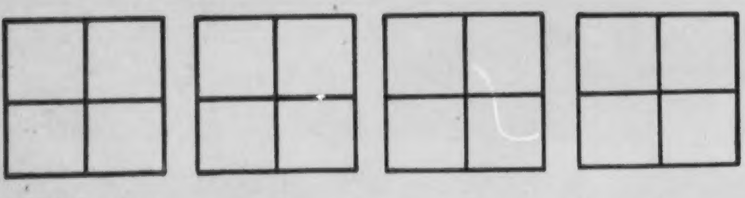

= ______


= ______


= ______


= ______

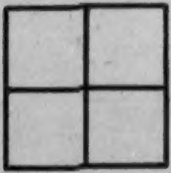

= ______

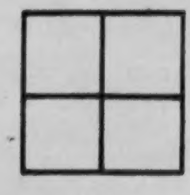

= ______

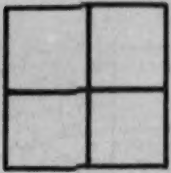
Name ______________________________

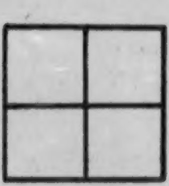
N 02 **

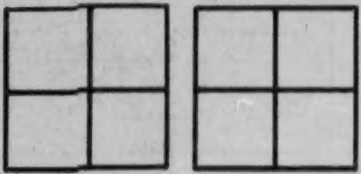
Put each of these numbers on the Minicomputer.

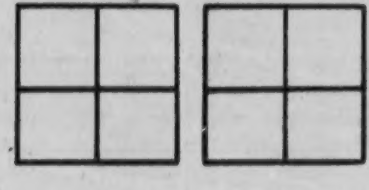

= 11

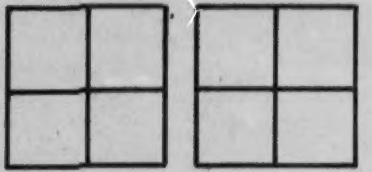

= 17

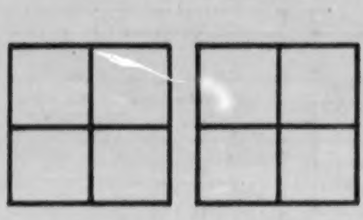

= 28

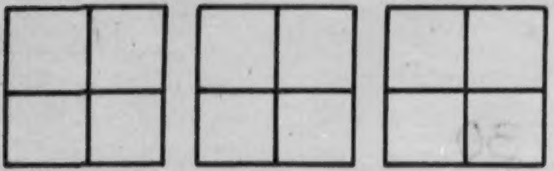

= 40

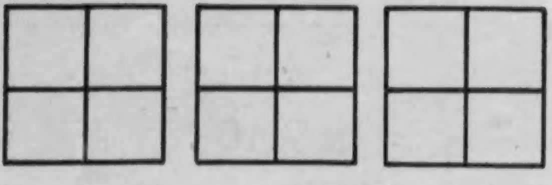

= 75


= 110


= 150


= 240

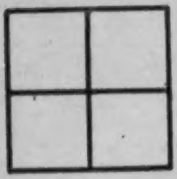

= 1,075

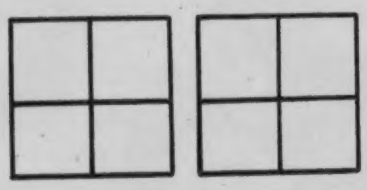

= 1,670

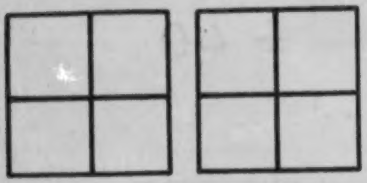
N 03 ***

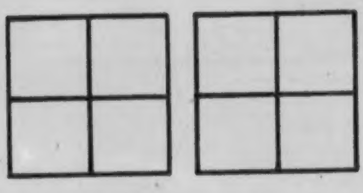
Name ____________________

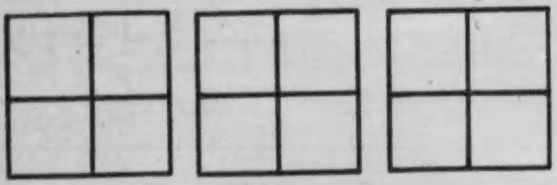
Write each of these numbers using exactly three checkers (positive or negative).

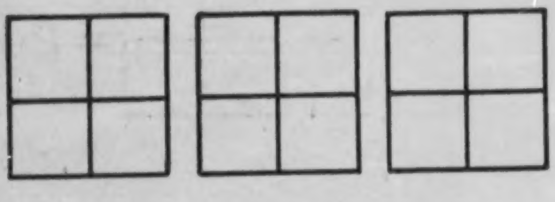

= 15

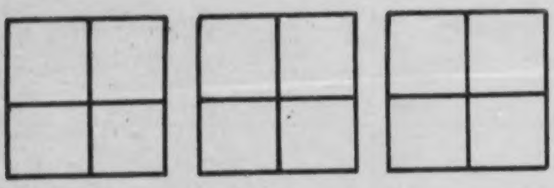

= 37

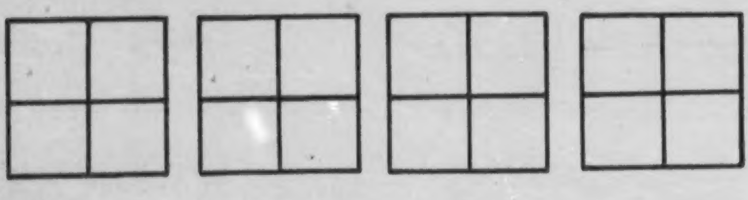

= 73


= 87


= 193


= 996


= 730

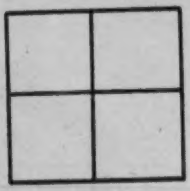

= 3,930

N 04 *

Name ______________________________

First clue

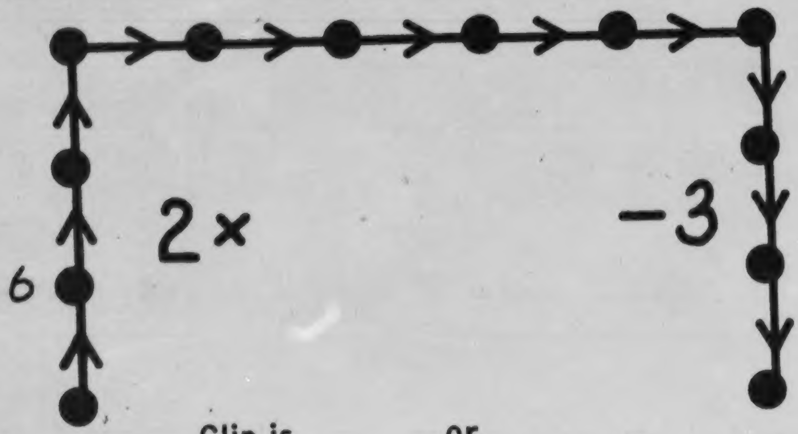
Clip is a secret number larger than 5 and can be put on this Minicomputer board using exactly two positive checkers.



Clip is ______ or ______ or ______ or ______ or ______ or ______.

Second clue

Clip is in this arrow picture.



Clip is ______ or ______.

Third clue

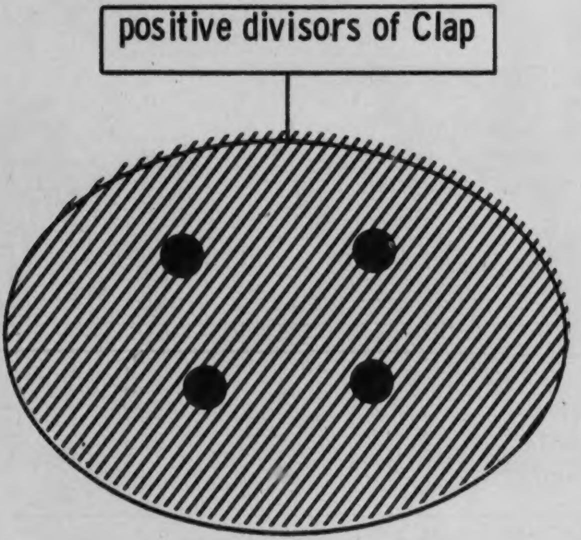
Clip has exactly four positive divisors.

Clip is ______.

Name ______________________

N 05 **

First clue

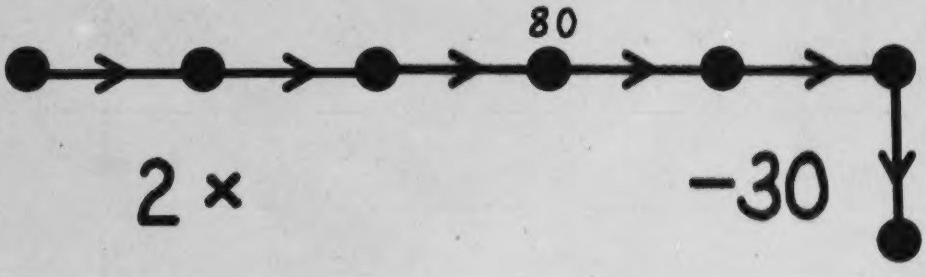

positive divisors of Clap

Clap could be _____; _____; _____; _____; _____; _____; _____;

_____; and so on.

Second clue

Clap is in this arrow picture.



Clap is _____.

N 06 *

Name ____________________

2 × 5 = ________

2 × 8 = ________

2 × 10 = ________

3 × 5 = ________

3 × 7 = ________

3 × 10 = ________

4 × 3 = ________

4 × 4 = ________

4 × 6 = ________

4 × 9 = ________

5 × 3 = ________

5 × 5 = ________

5 × 8 = ________

2 × ☐ = 14

2 × ☐ = 18

3 × ☐ = 9

3 × ☐ = 18

4 × ☐ = 8

4 × ☐ = 20

4 × ☐ = 40

5 × ☐ = 10

5 × ☐ = 20

5 × ☐ = 30

6 × ☐ = 12

6 × ☐ = 24

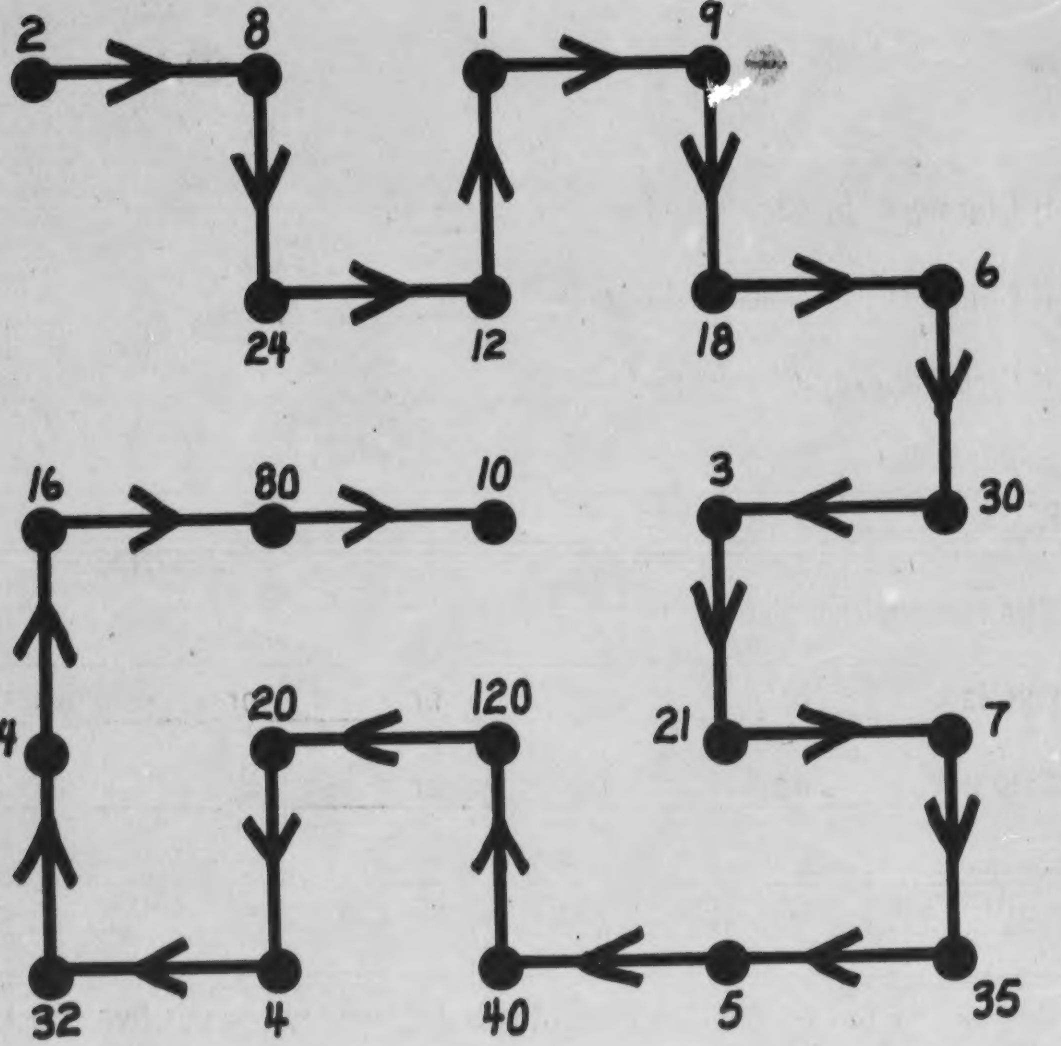
6 × ☐ = 48

7 × ☐ = 21

Name ______________________

N 07 **

Label each arrow "x ☐" or "÷ ☐".

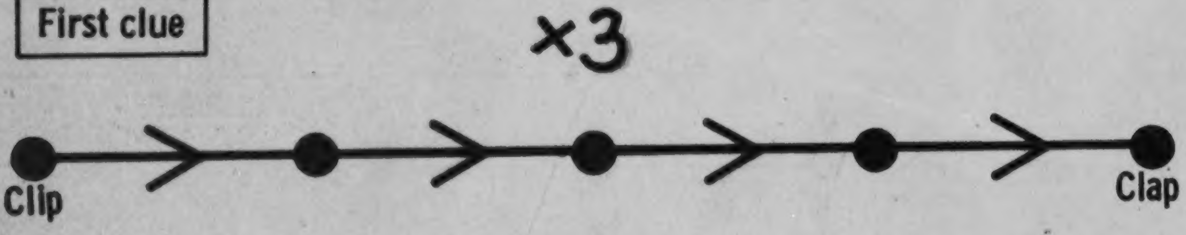

2
8
1
9
24
12
18
6
16
80
10
3
30
20
120
21
7
64
32
4
40
5
35

Name ______________________________

N 08 ***

Clip and Clap are two secret numbers.

First clue



If Clip were 5, Clap would be __________.

If Clip were 7, Clap would be __________.

If Clap were 81, Clip would be __________.

Second clue

Clip is a positive divisor of 12.

Clip is _______ or _______ or _______ or _______ or _______ or _______.

Clap is _______ or _______ or _______ or _______ or _______ or _______.

Third clue

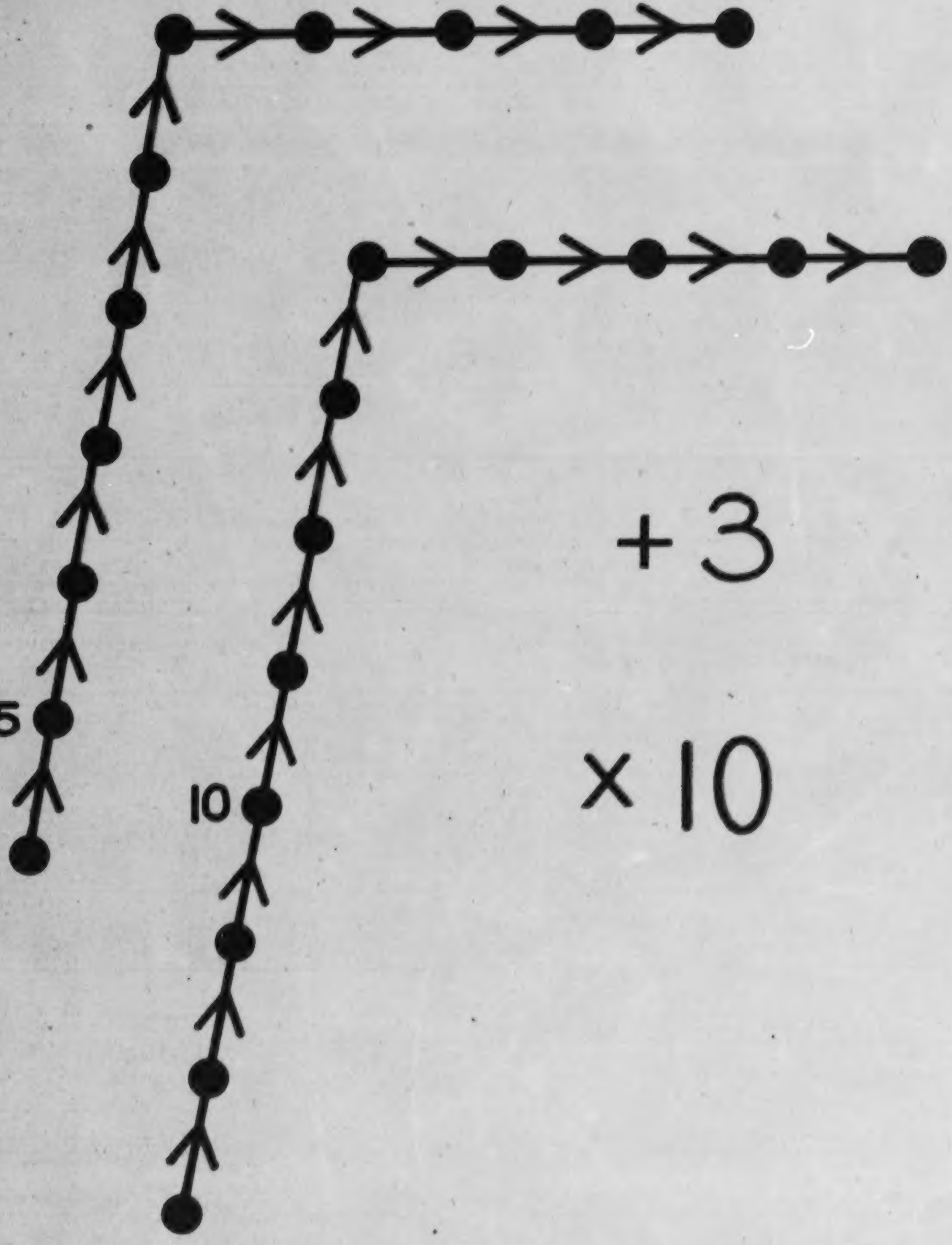
Clap can be put on the Minicomputer using exactly one positive checker and two negative checkers.

Clap is __________. Clip is __________.

N 09 *

Name ____________________

Label the dots.



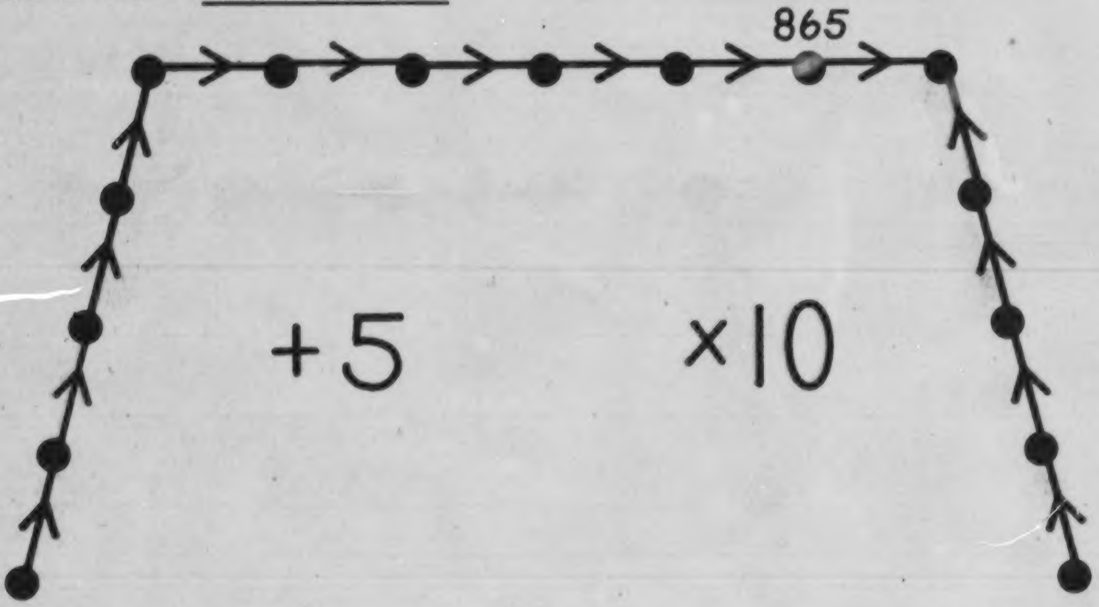
Name ______________________________

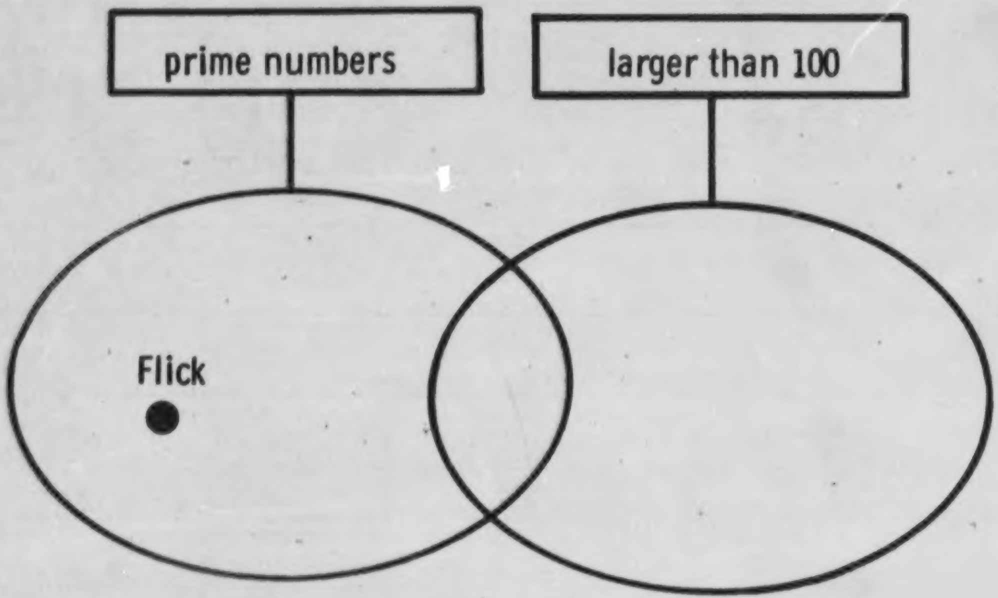
N 10 **

Flick is a secret odd number.

Flick is in this arrow picture and in this string picture.

Who is Flick ? ____________



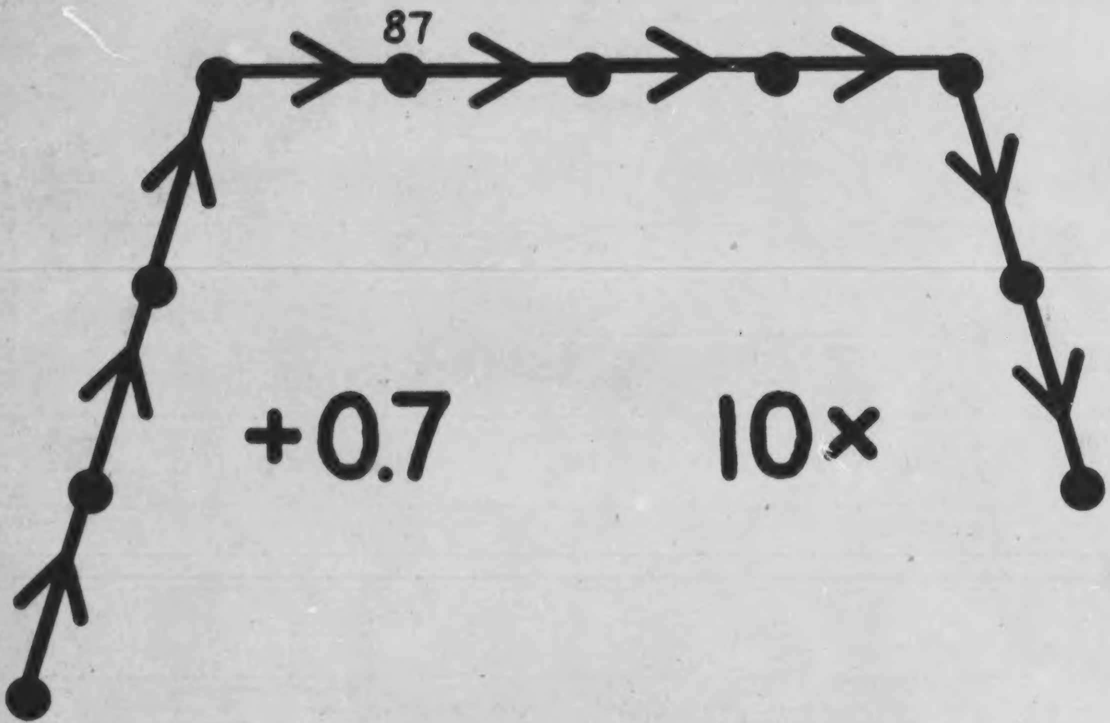

prime numbers
larger than 100
Flick

Name ______________________________

N 11 ***

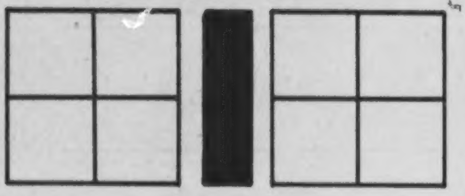
Flake is a secret number.

Flake is in this arrow picture.



Flake can be put on this Minicomputer using exactly one checker (positive or negative) on each board.

Flake =

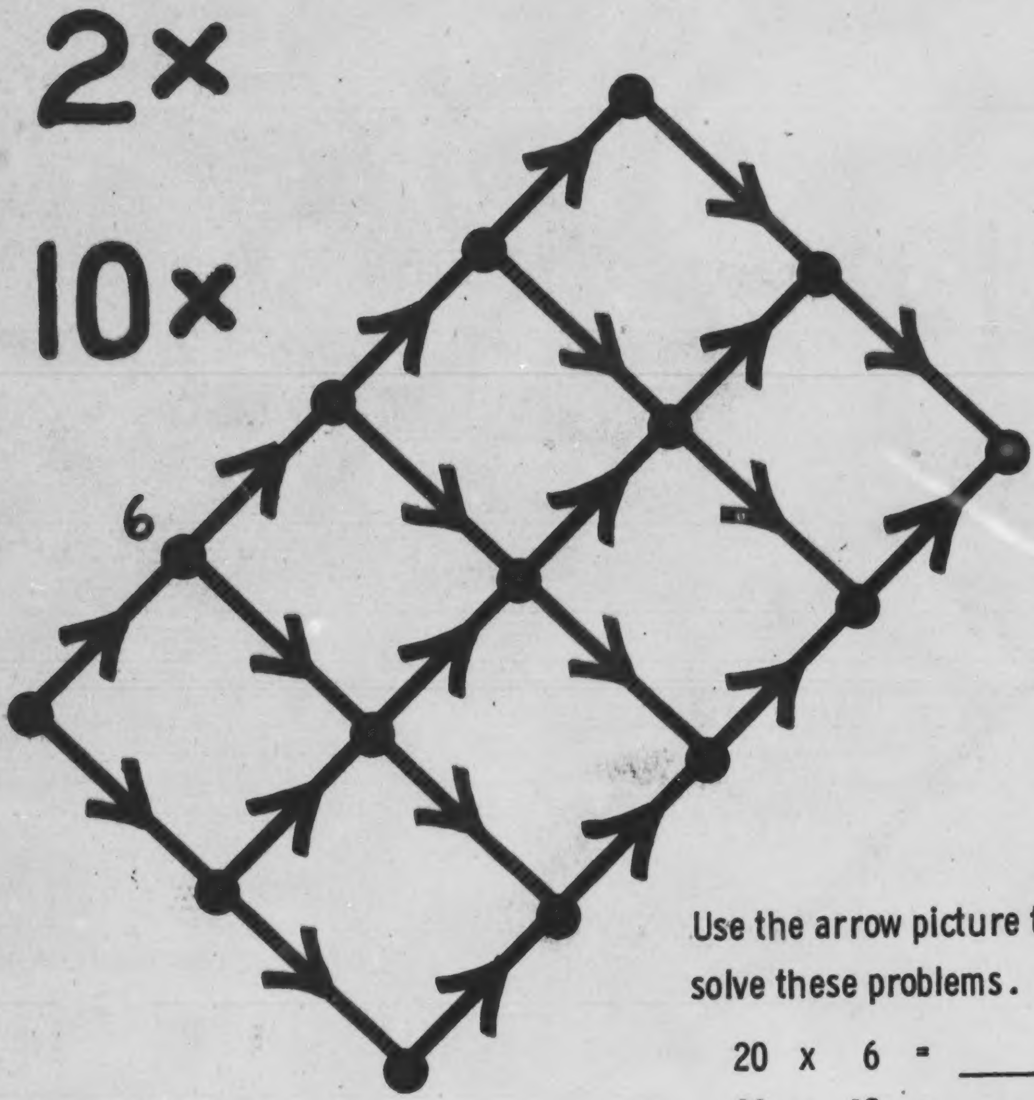


= ____________.

N 12 *

Name ______________________

Label all the dots. Use green to draw as many 20x arrows as you can.



Use the arrow picture to solve these problems.

20 x 6 = ________

20 x 12 = ________

20 x 24 = ________

60 ÷ 10 = ________

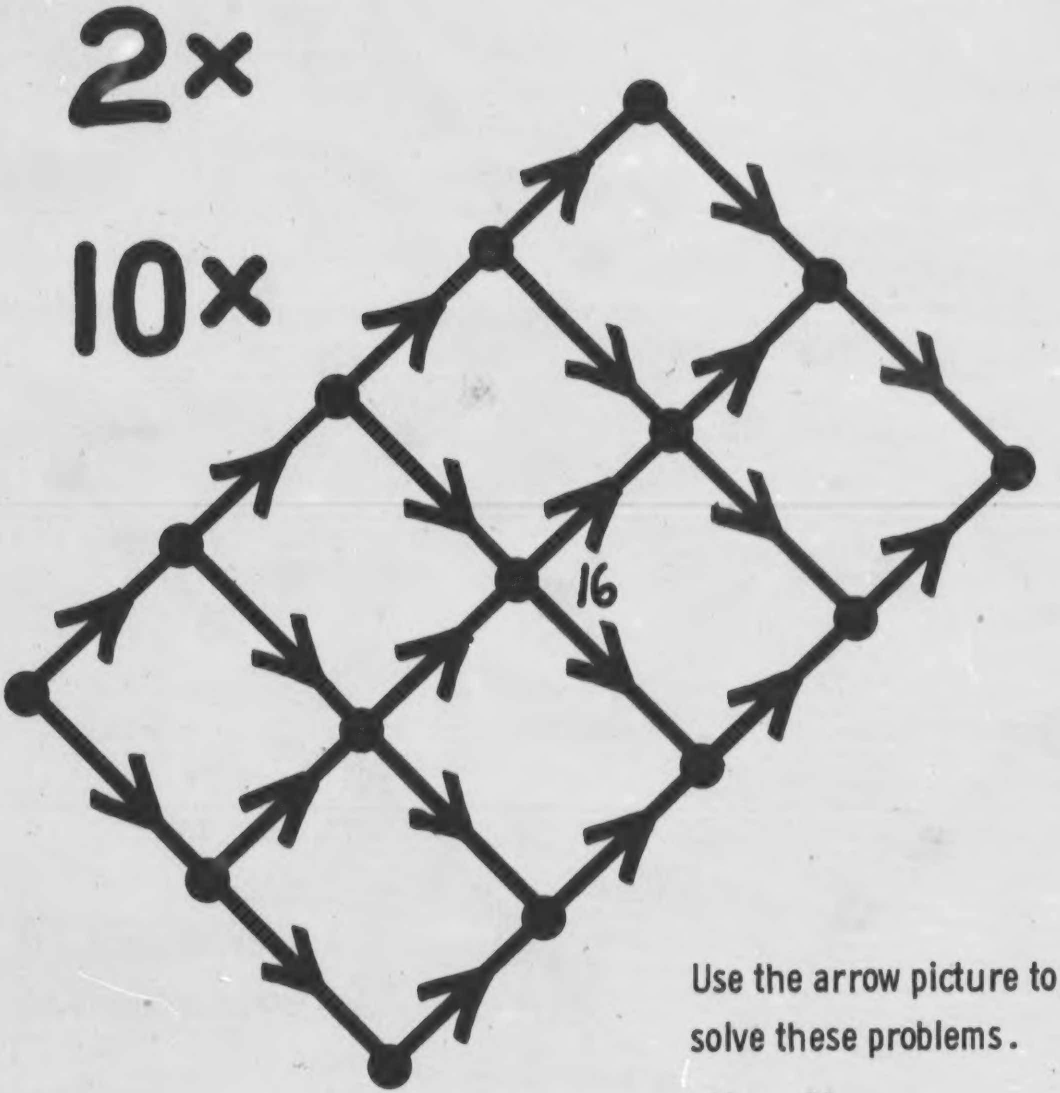
120 ÷ 10 = ________

1,200 ÷ 10 = ________

Name ________________________

N 13 **

Label all the dots. Use green to draw as many 20x arrows as you can.



Use the arrow picture to solve these problems.

32 ÷ 20 = ________

4 ÷ 10 = ________

160 ÷ 20 = ________

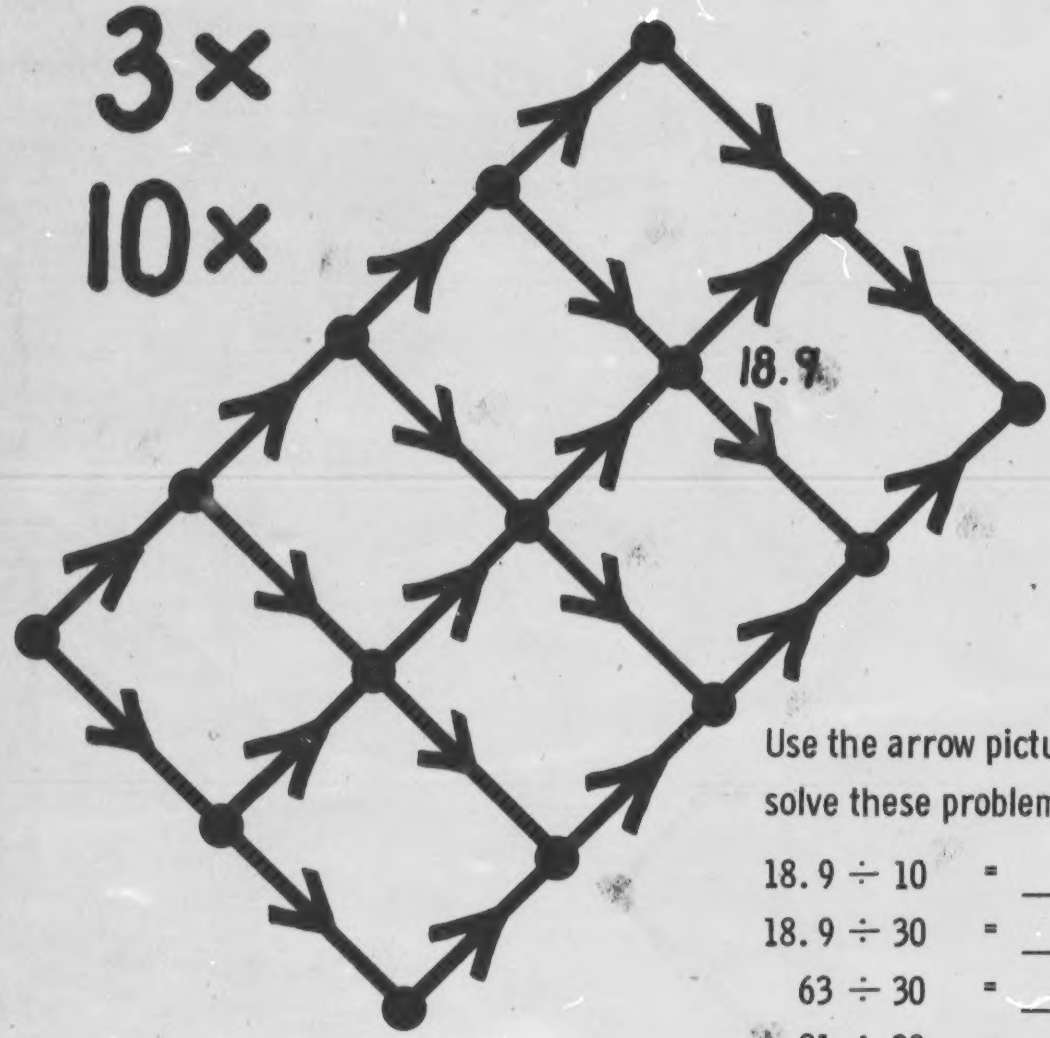
8 ÷ 20 = ________

64 ÷ 20 = ________

N 14 ***

Name __________________________

Label all the dots. Use green to draw as many 30x arrows as you can.



Use the arrow picture to solve these problems.

18.9 ÷ 10 = ________

18.9 ÷ 30 = ________

63 ÷ 30 = ________

21 ÷ 30 = ________

9 × 0.21 = ________

6.3 ÷ 9 = ________

0.63 ÷ 3 = ________

0.63 ÷ 9 = ________

N 15 *

Name ____________________

$$\begin{array}{r} 375 \\ \times\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 2,344 \\ \times\ 3 \\ \hline \end{array}$$

$$\begin{array}{r} 2,652 \\ \times\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 3,215 \\ \times\ 3 \\ \hline \end{array}$$

10 x 25 = ____________

2 x 25 = ____________

12 x 25 = ____________

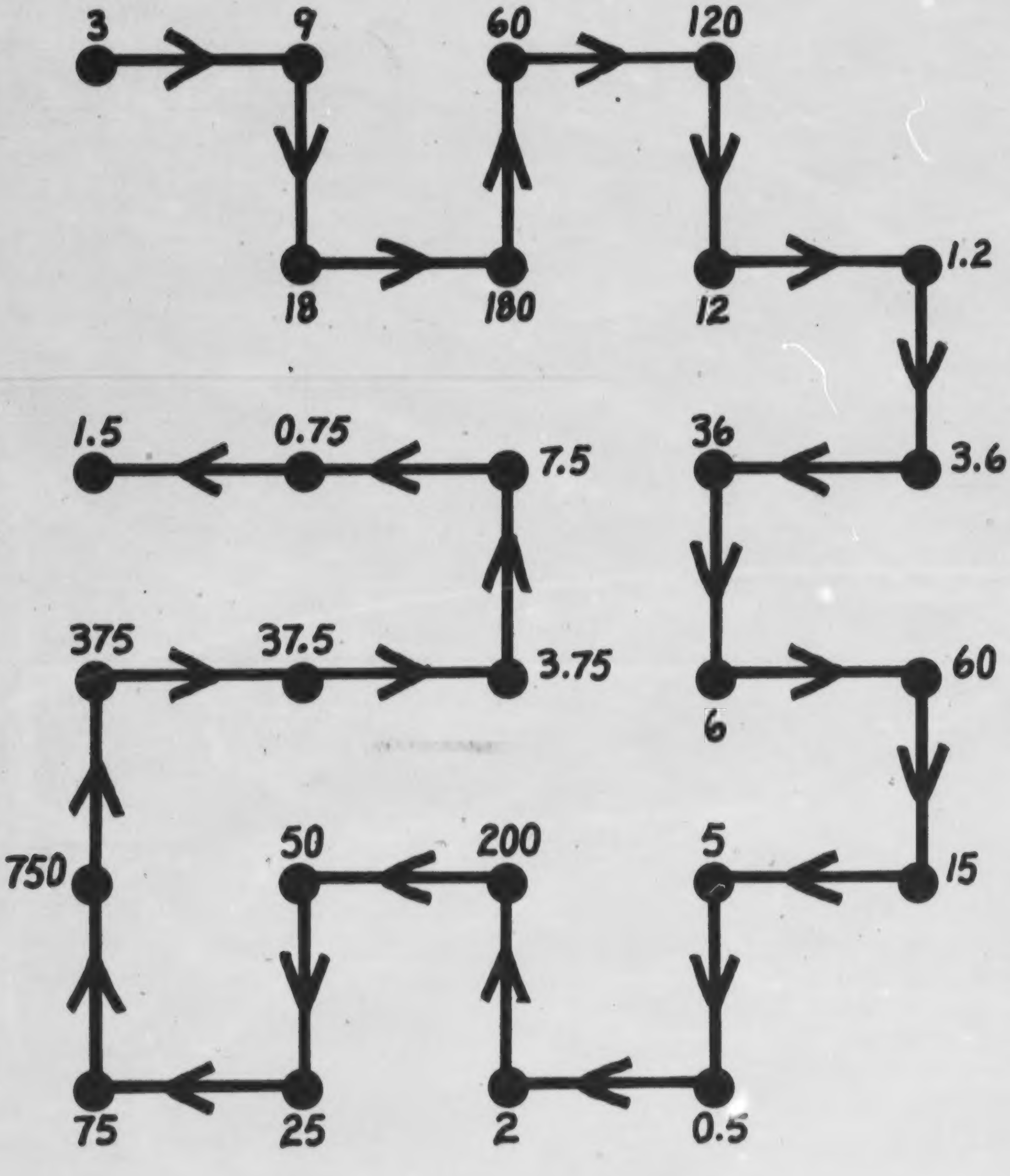
20 x 25 = ____________

22 x 25 = ____________

Name ______________________

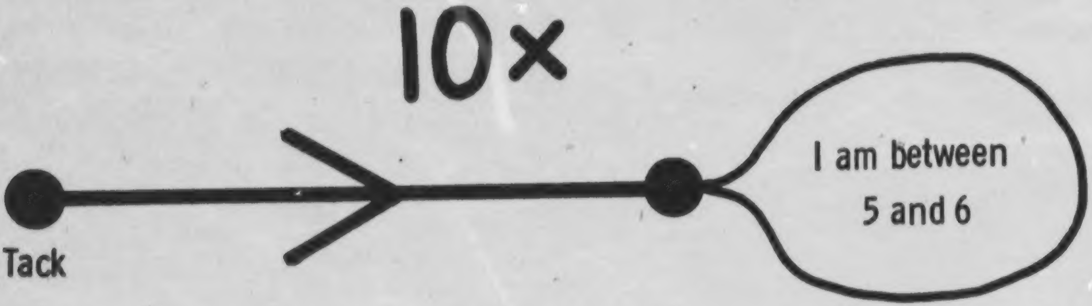
N 16 **

Label each arrow "☐ X" or " ÷ ☐ ".


3
9
60
120
18
180
12
1.2
1.5
0.75
7.5
36
3.6
375
37.5
3.75
6
60
750
50
200
5
15
75
25
2
0.5

N 17 ***

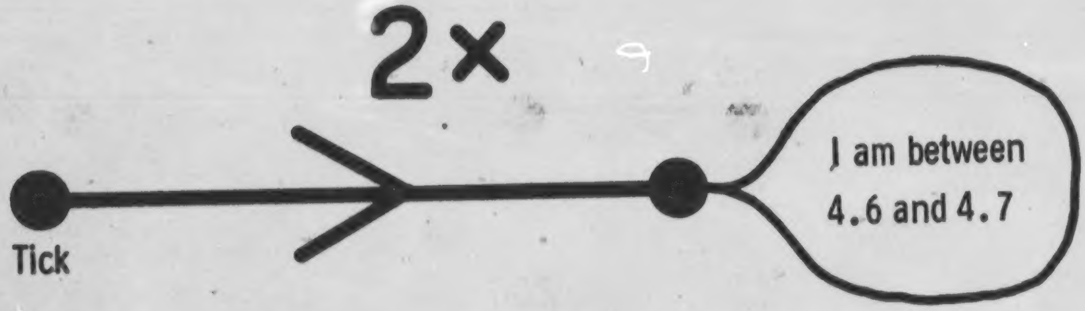
Name ________________________________


10×
Tack
I am between 5 and 6

Tack could be ________; ________; ________; ________; ________;

________; ________; ________; ________; ________; ________;

________; and so on.


2×
Tick
I am between 4.6 and 4.7

Tick could be ________; ________; ________; ________; ________;

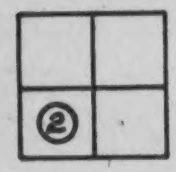
________; ________; ________; ________; ________; ________;

________; ________; ________; and so on.

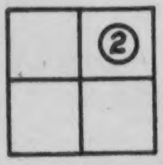
N 18 *

Name ______________________

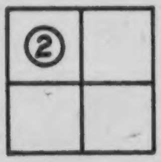
What number is written on the Minicomputer ?



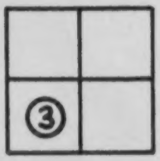
______



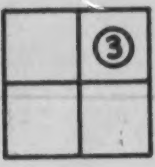
______



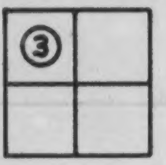
______



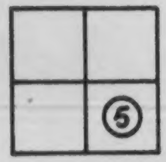
______



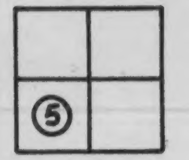
______



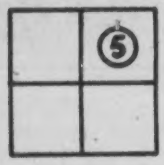
______



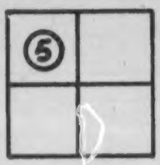
______



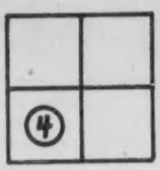
______



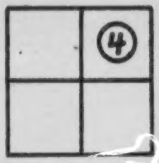
______



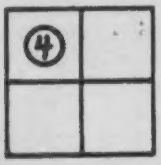
______



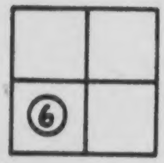
______



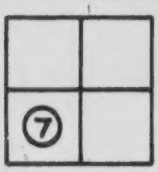
______



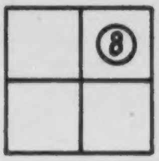
______



______



______



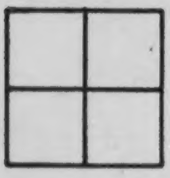
______

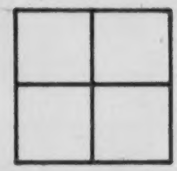
Name ______________________

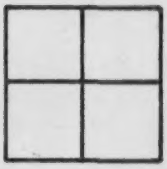
N 19 **

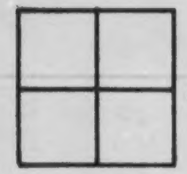
Put these numbers on the Minicomputer using exactly one of these checkers:

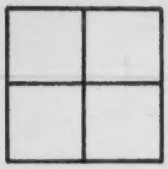
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

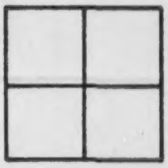

= 12

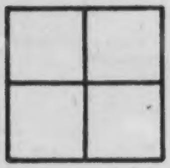

= 10

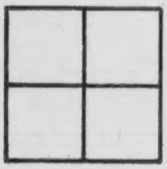

= 18

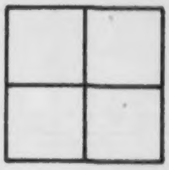

= 16

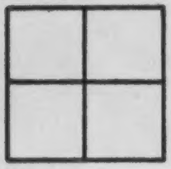

= 20

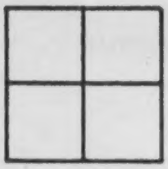

= 14

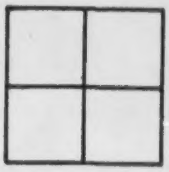

= 28

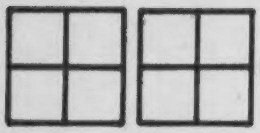

= 24

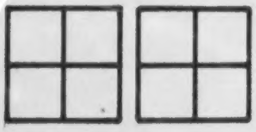

= 48

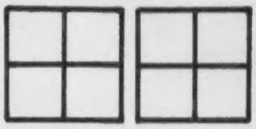

= 56


= 40


= 72


= 100


= 400


= 280

Name ______________________

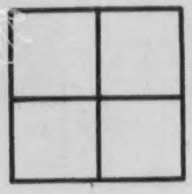
N 20 ***

Diz is a secret number.

First clue

Diz is a multiple of 6 and can be put on this Minicomputer using exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



Diz is _____ or _____ or _____ or _____ or _____ or _____ or _____.

Second clue

If you add 1 to Diz, you will get a square number.

Diz is __________ or __________.

Third clue

Diz has exactly ten positive divisors.

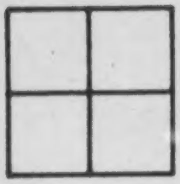
Diz is ____________.

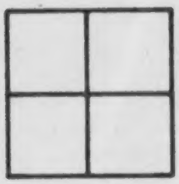
N 21 ****

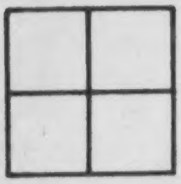
Name ____________________

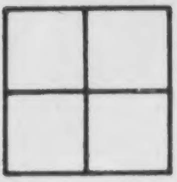
Put each of these numbers on the Minicomputer using exactly two of these checkers:

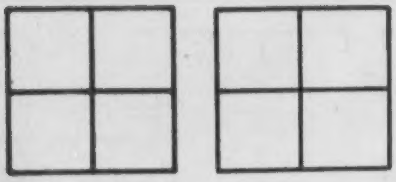
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

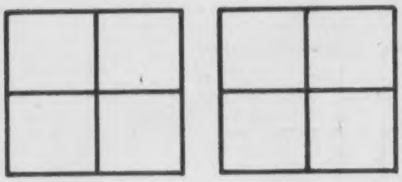

= 50

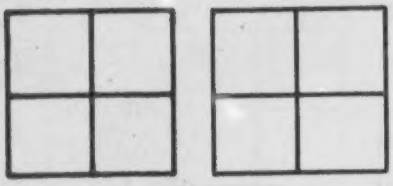

= 60

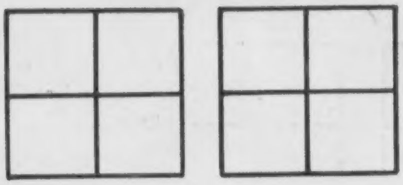

= 100


= 75


= 250


= 370


= 792

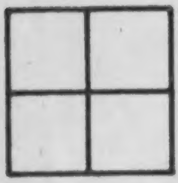

= 1,200

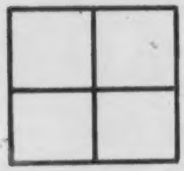
Name ______________________________

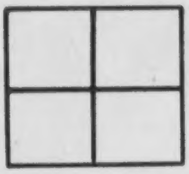
N 22 *

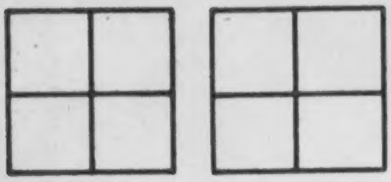
Write any number you wish on the Minicomputer using exactly two of these checkers:

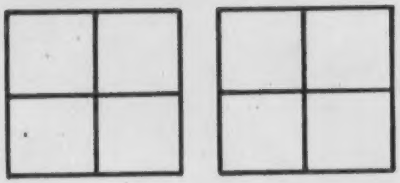
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

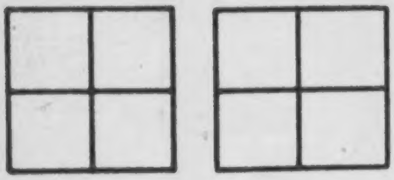

= ______

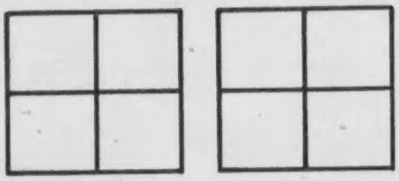

= ______

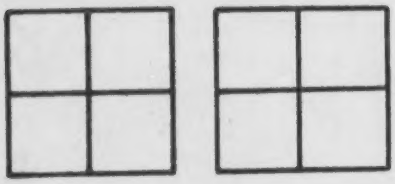

= ______

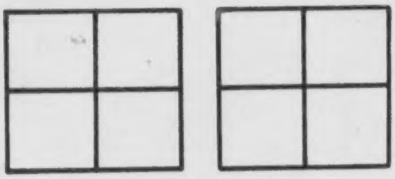

= ______


= ______


= ______


= ______


= ______


= ______

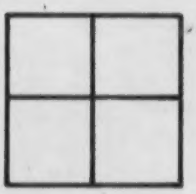
Name ______________________

N 23 **

First clue

Cri is a multiple of 4 and can be put on this Minicomputer board using exactly one of these checkers:

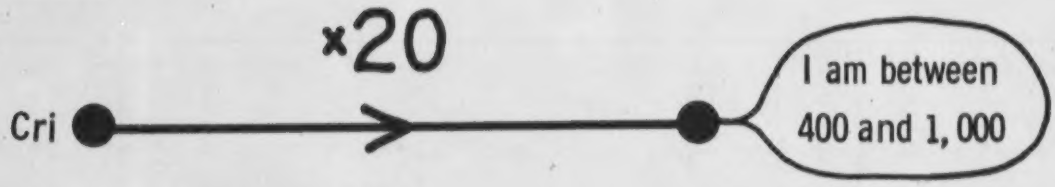
② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



Cri is _____ or _____ or _____ or _____ or _____ or _____ or _____

or _____ or _____ or _____ or _____ or _____ or _____ or _____.

Second clue


×20
Cri
I am between 400 and 1,000

Cri is ______ or ______ or ______ or ______ or ______ or ______.

Third clue

Cri is a square number.

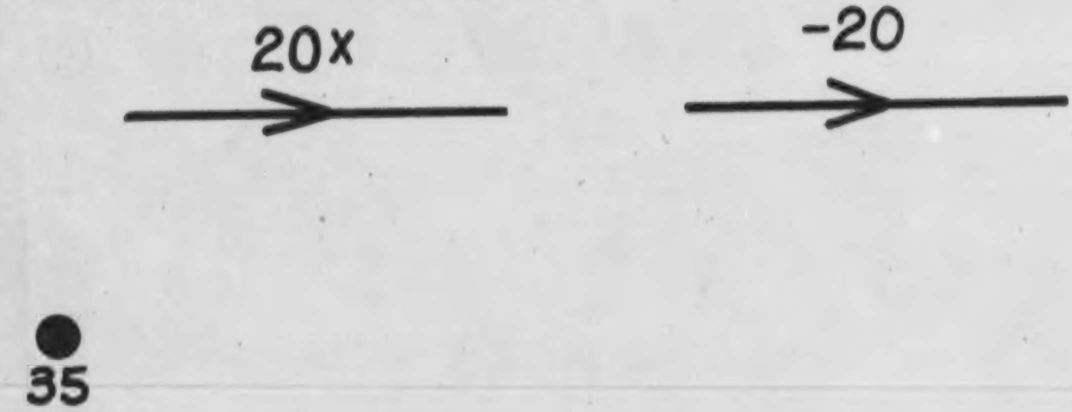
Cri is ______.

N 24 ***

Name ______________________________

First clue

Tack is the ending number of a red-blue road starting at 35 and using exactly two red arrows and two blue arrows.



Tack is ______ or ______ or ______ or ______ or ______ or ______.

Second clue

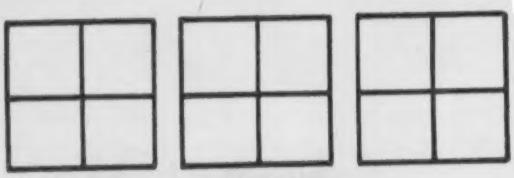
Tack is a multiple of 7.

Tack is ______ or ______.

Third clue

Tack can be put on this Minicomputer using exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

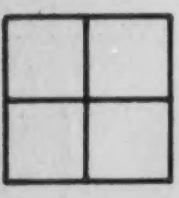
Tack =  = __________.

Name ______________________

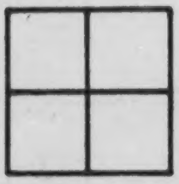
N 25 ,*

Write any number you wish on the Minicomputer using exactly two of these checkers:

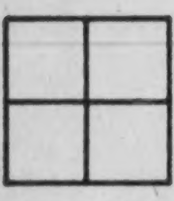
③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



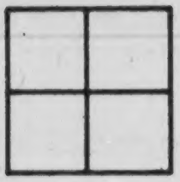
= ______



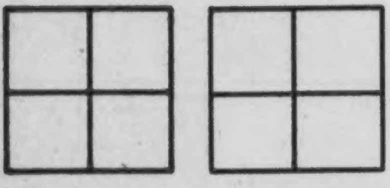
= ______



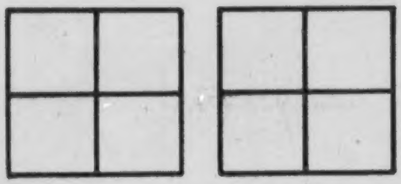
= ______



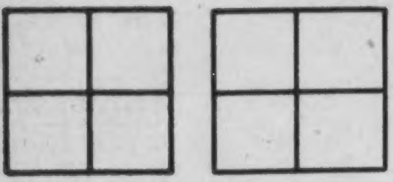
= ______



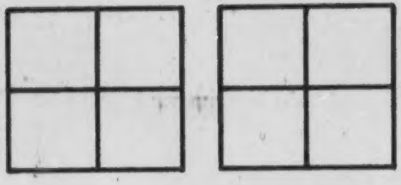
= ______



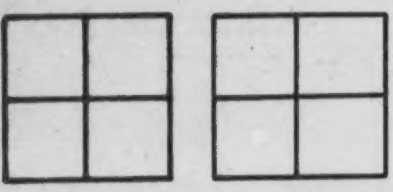
= ______



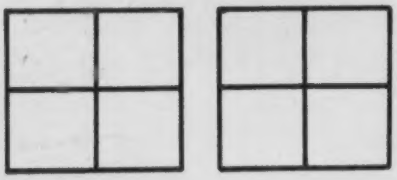
= ______



= ______



= ______



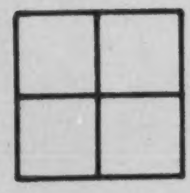
= ______

Name ______________________________

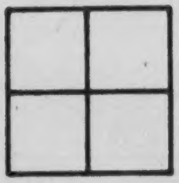
N 26 **

Write each number on the Minicomputer using exactly these two checkers:

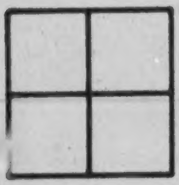
② ⑤



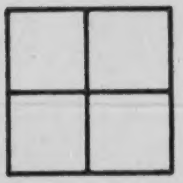
= 18



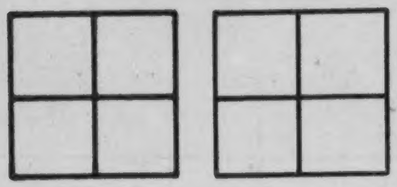
= 22



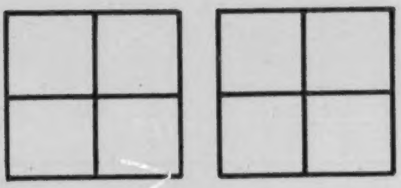
= 44



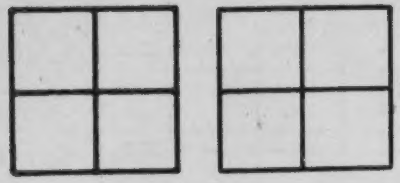
= 36



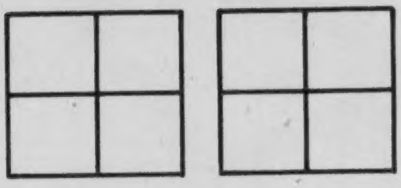
= 108



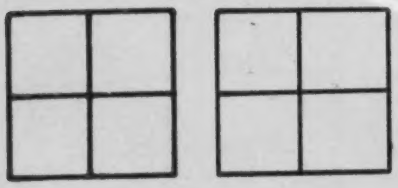
= 240



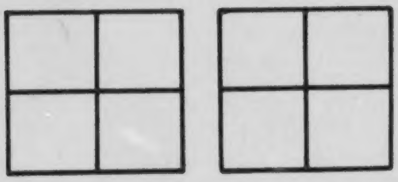
= 66



= 480



= 70



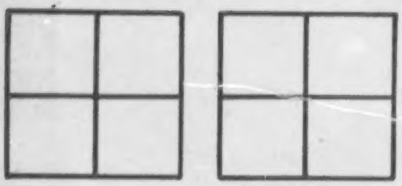
= 560

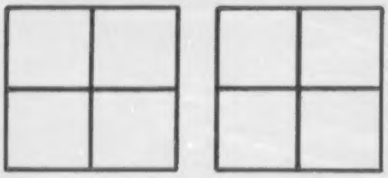
Name ______________________________

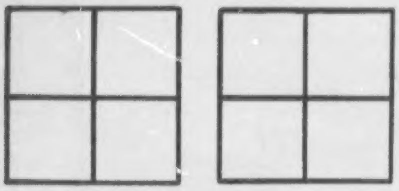
N 27 ***

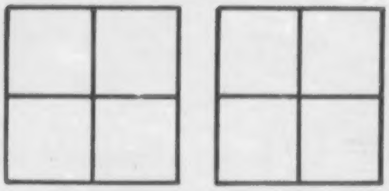
Write each number on the Minicomputer using exactly one positive regular checker and all of these three new checkers.

● ② ③ ④


= 100

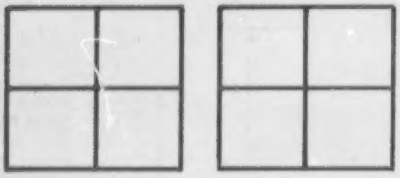

= 200

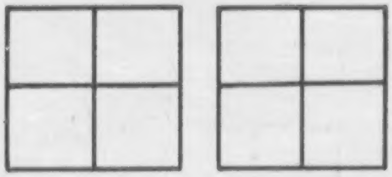

= 125

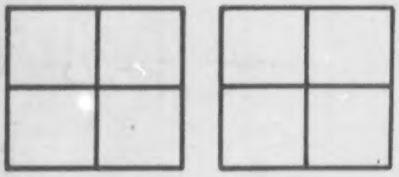

= 350

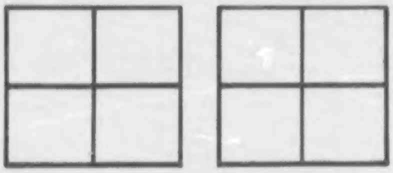
Write each number on the Minicomputer using exactly one positive regular checker and all of these five new checkers.

● ② ③ ④ ⑤ ⑥


= 200


= 500


= 325


= 1,000

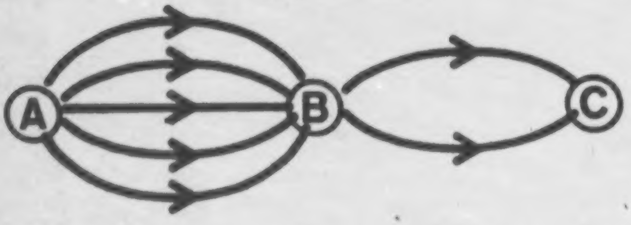
N 28 *

Name ______________________

How many roads can you choose to go . . .

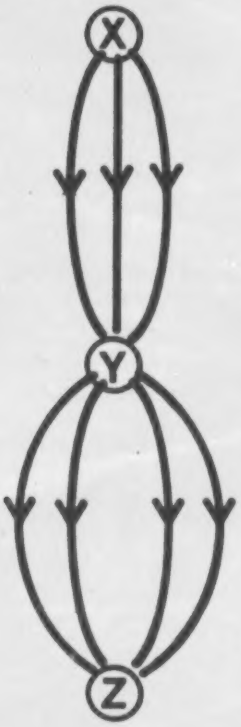
. . . from A to C?

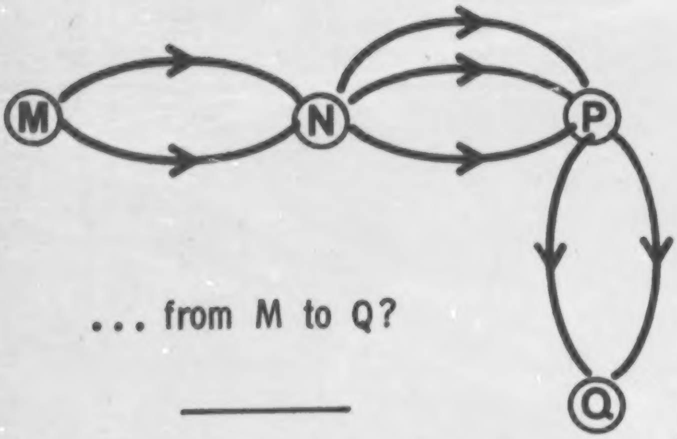
______



. . . from X to Z?

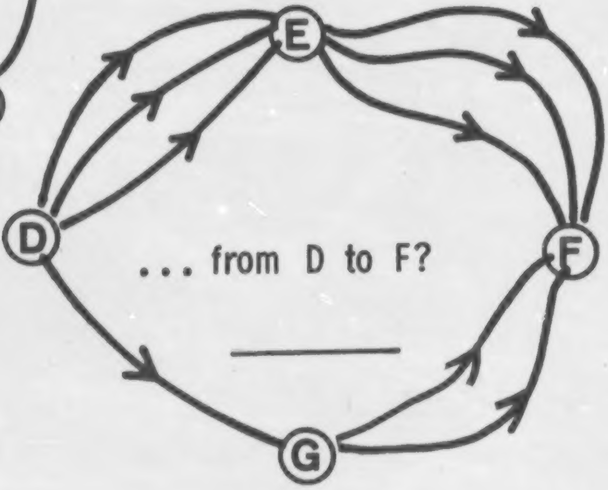
______





. . . from M to Q?

______



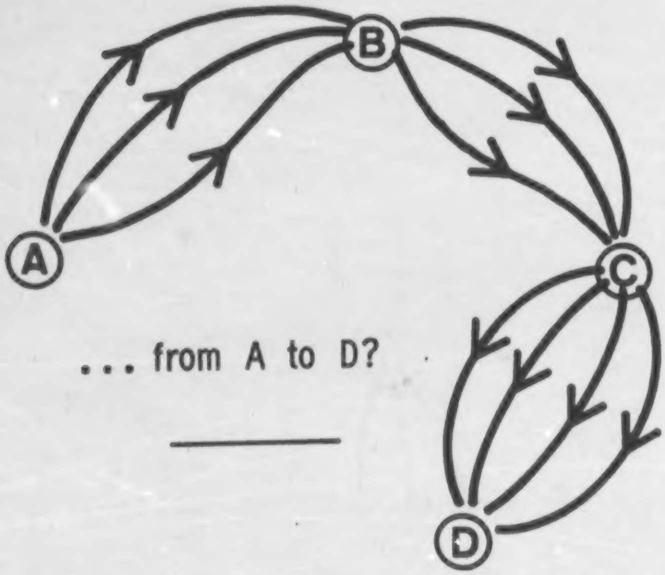
. . . from D to F?

______

N 29 **

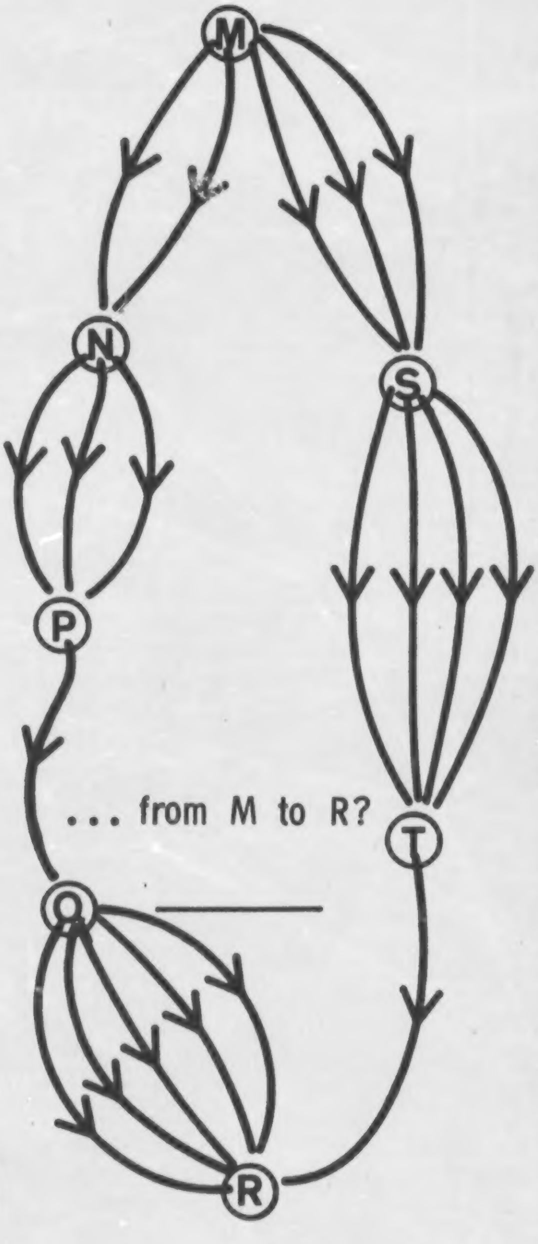
Name ______________________________

How many roads can you choose to go . . .



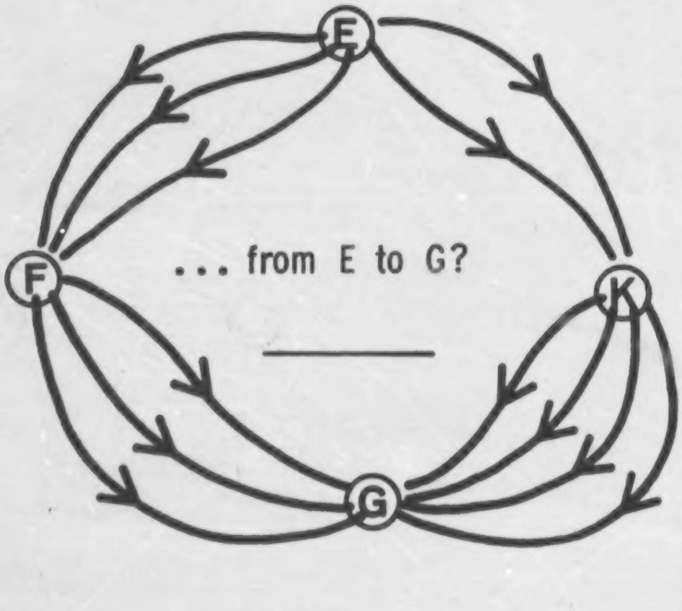
. . . from A to D?

____



. . . from M to R?

____



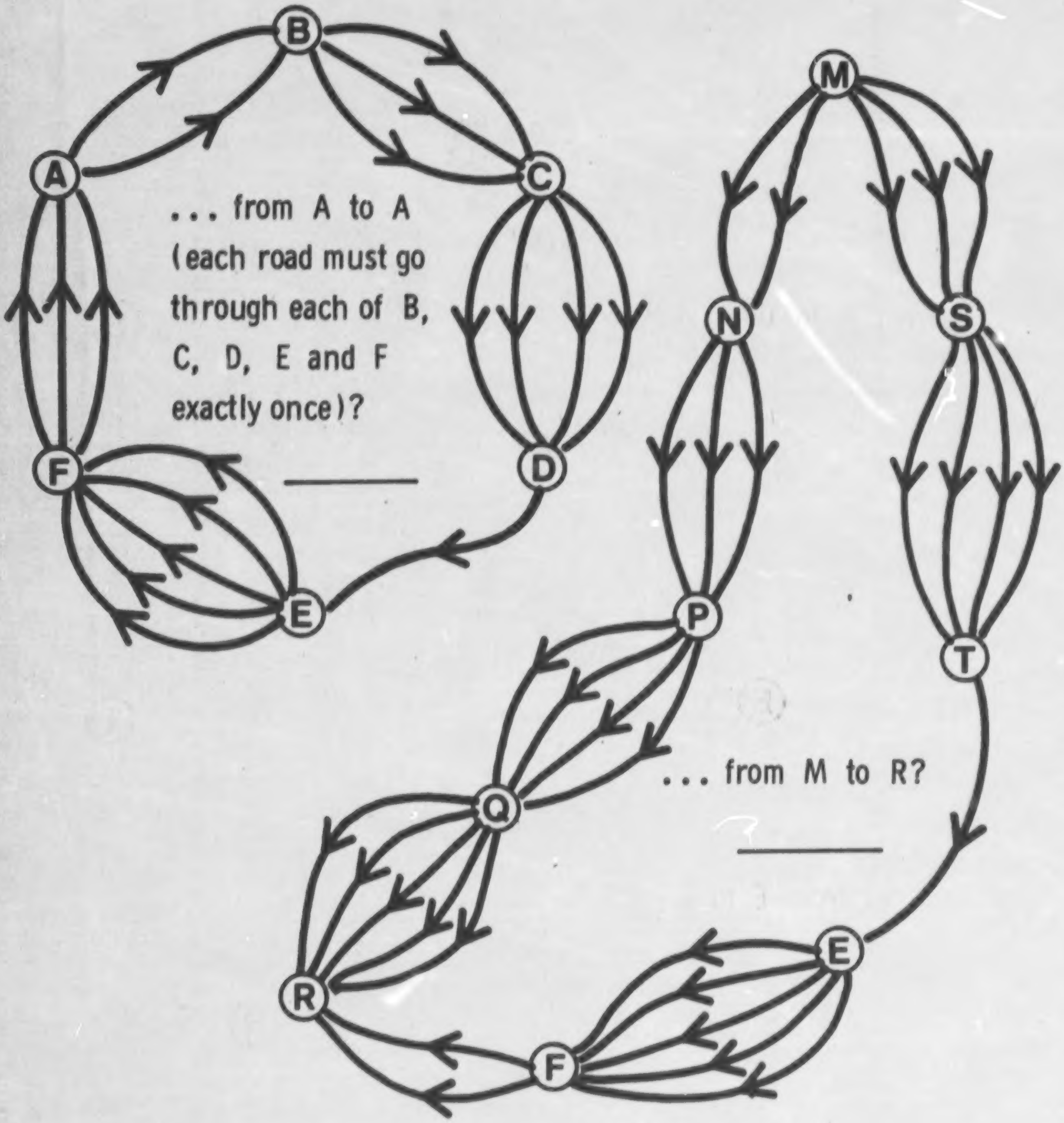
. . . from E to G?

____

Name ____________________

N 30 ***

How many roads can you choose to go . . .


A
B
C
D
E
F
M
N
S
P
T
Q
R
E
F

. . . from A to A (each road must go through each of B, C, D, E and F exactly once)?

____

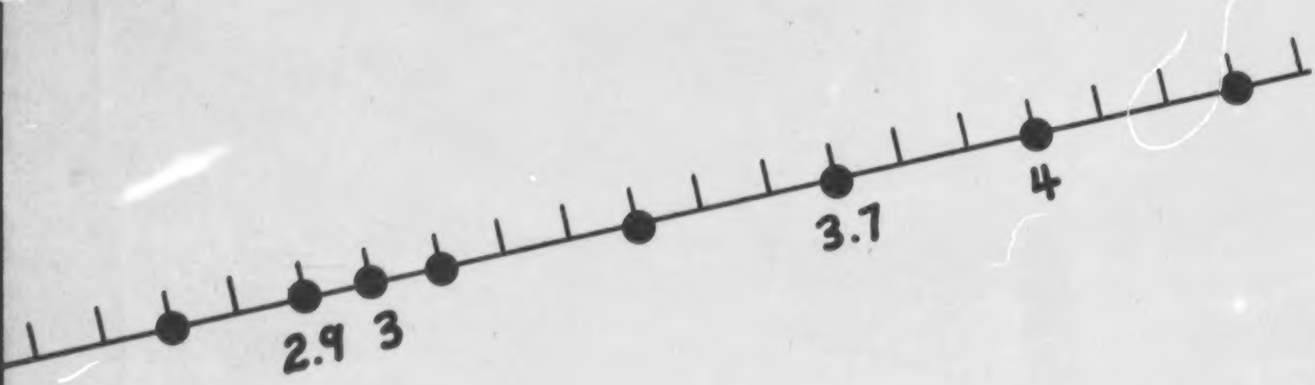
. . . from M to R?

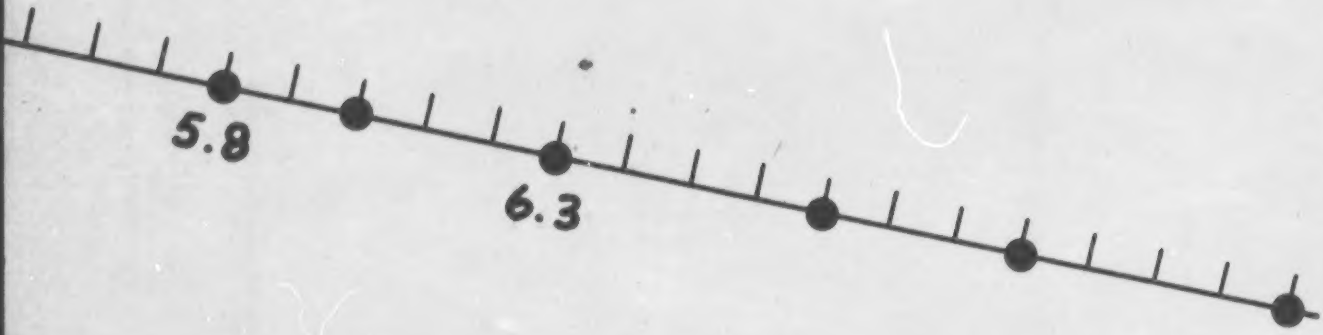
____

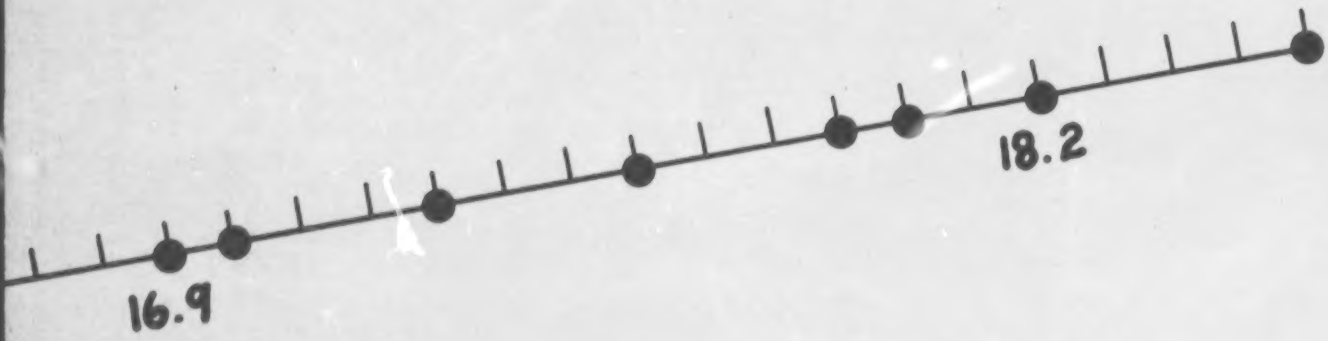
N 31 *

Name ______________________________

Label the dots.





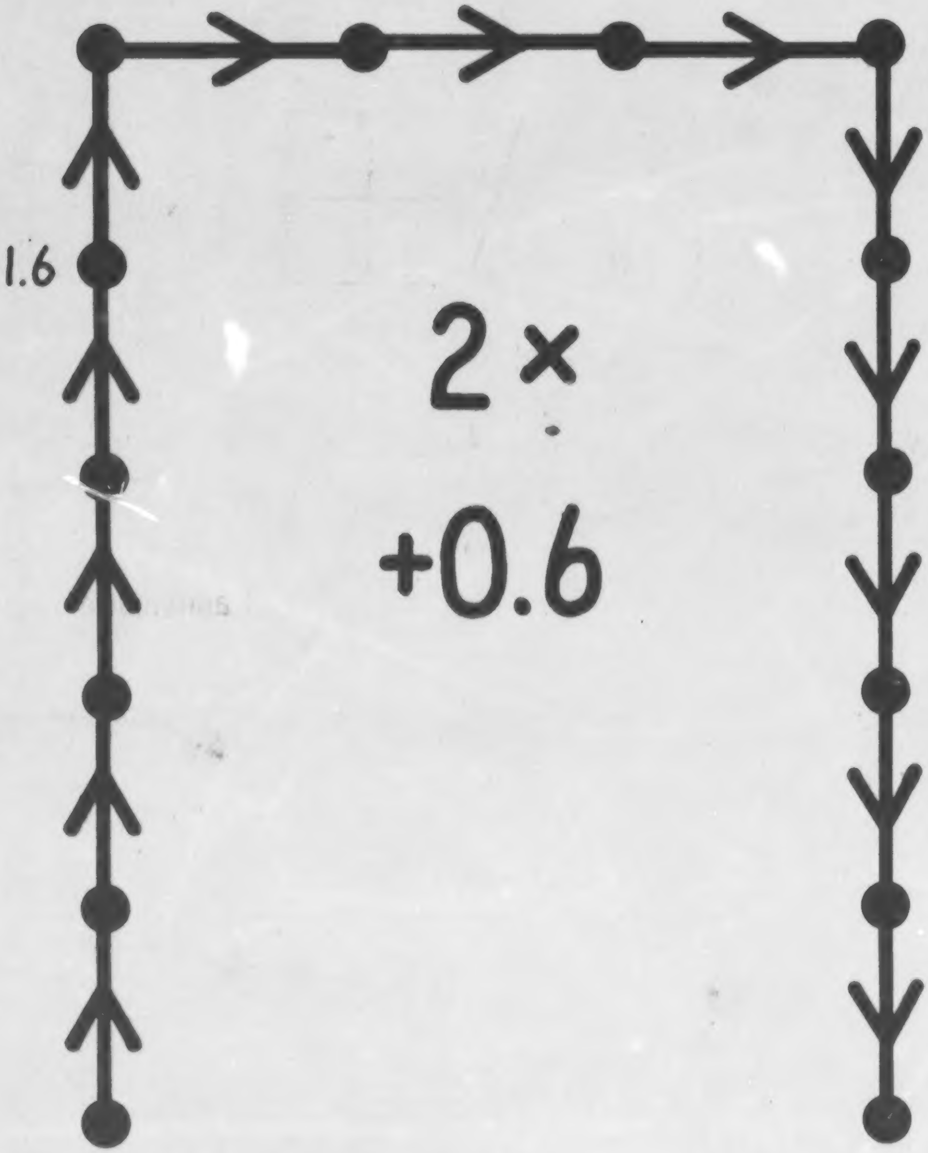


N 32 **

Name ______________________________

First clue

Flip is in this picture.

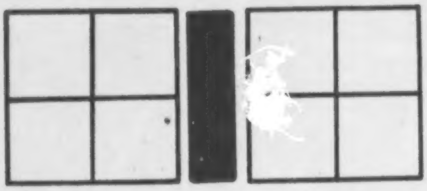


N 33 ***

Name ______________________________

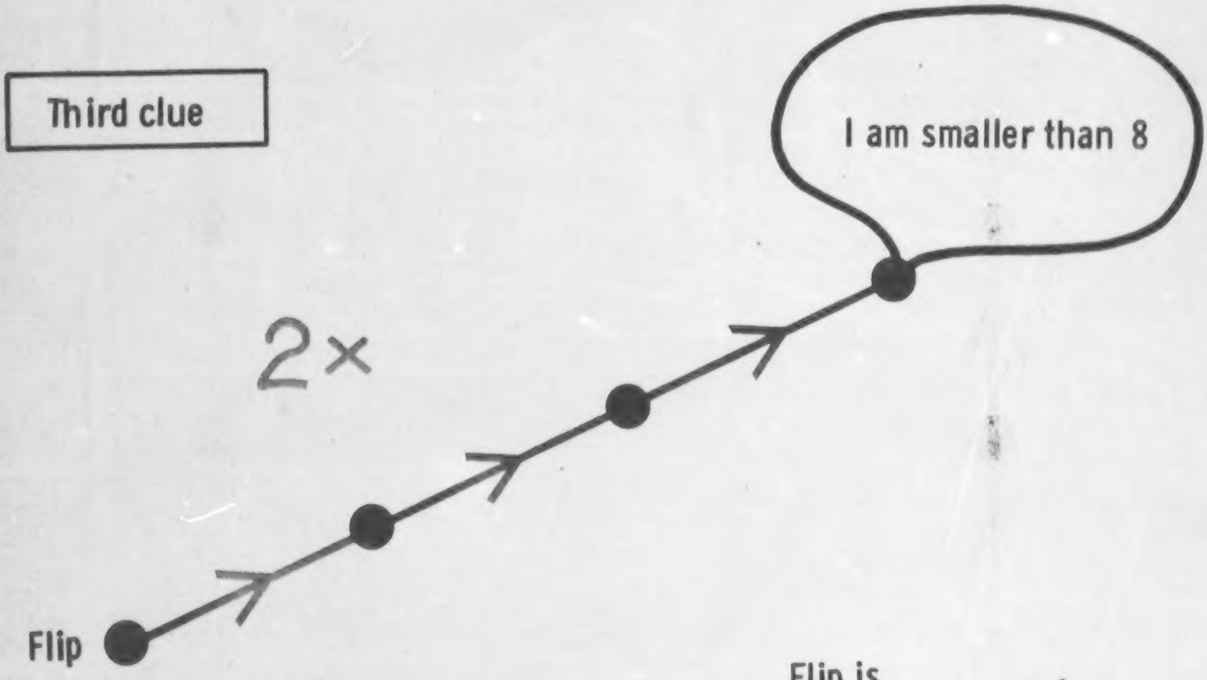
Second clue

Flip can be put on the Minicomputer using exactly one checker (positive or negative) on each board.



Flip could be _____ or _____ or _____.

Third clue


I am smaller than 8
2×
Flip

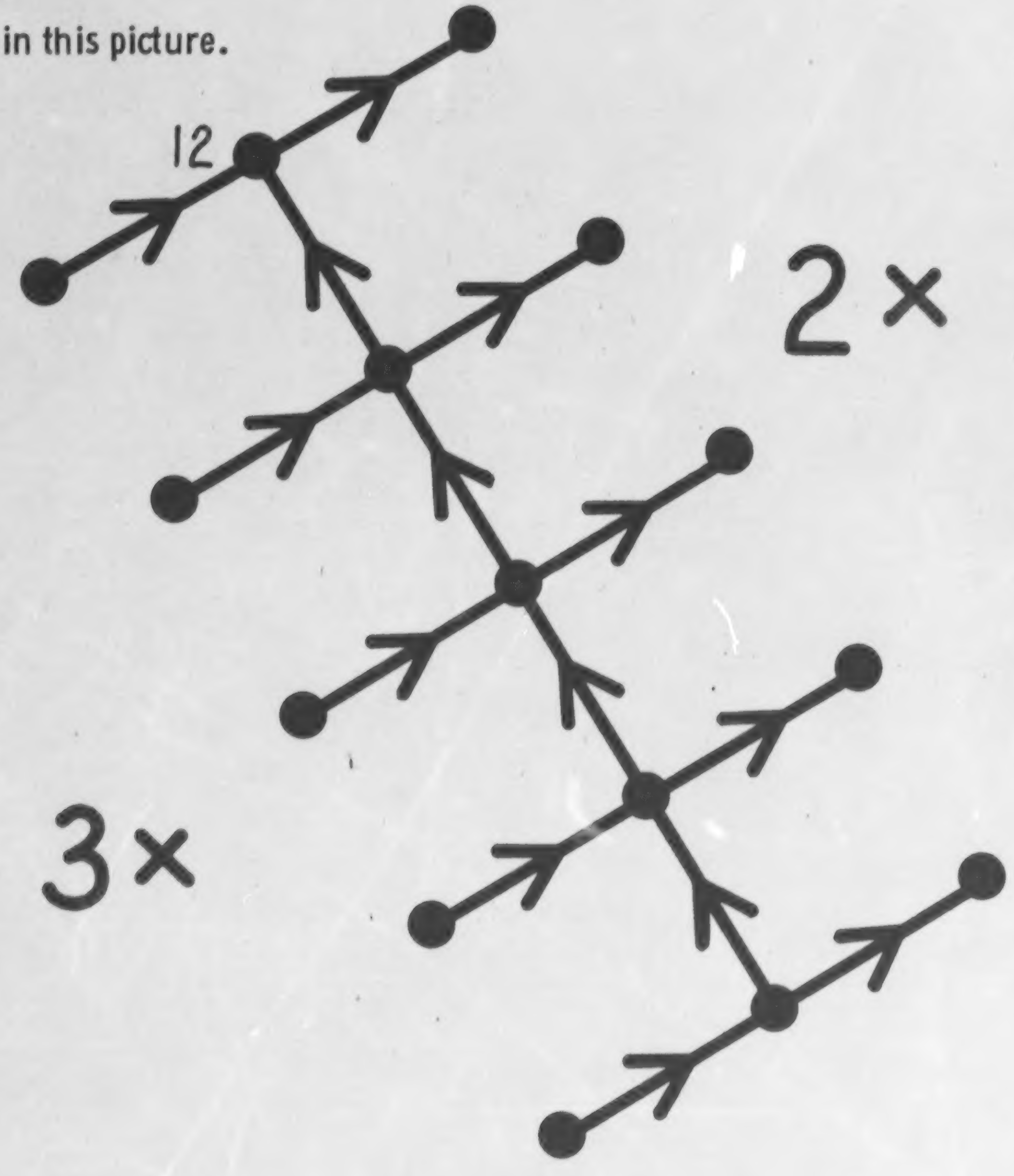
Flip is _______.

Name ______________________

N 34 *

Third clue

Zip is in this picture.



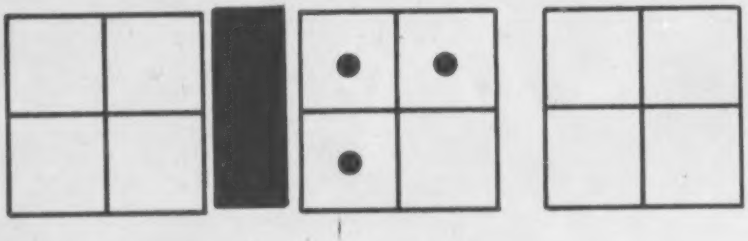
Zip could be _____ or _____ or _____ or _____ or _____.

N 35 **

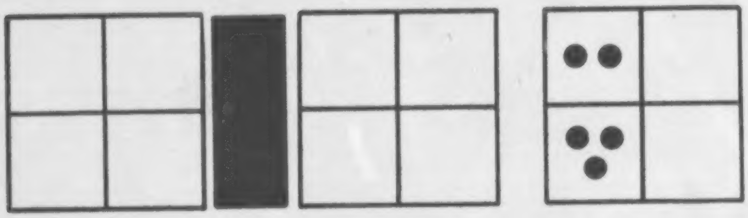
Name ____________________

Fourth clue

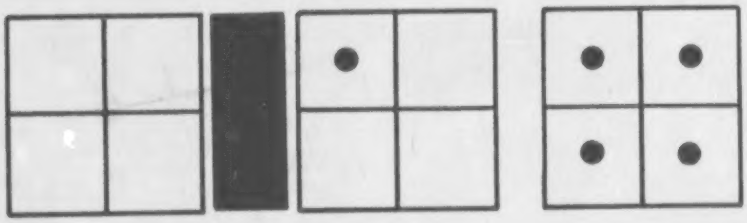
Zip is one of these numbers.



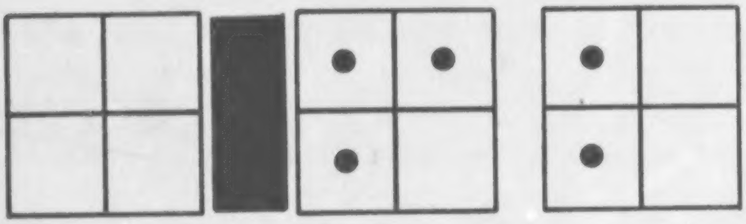
= ________



= ________



= ________



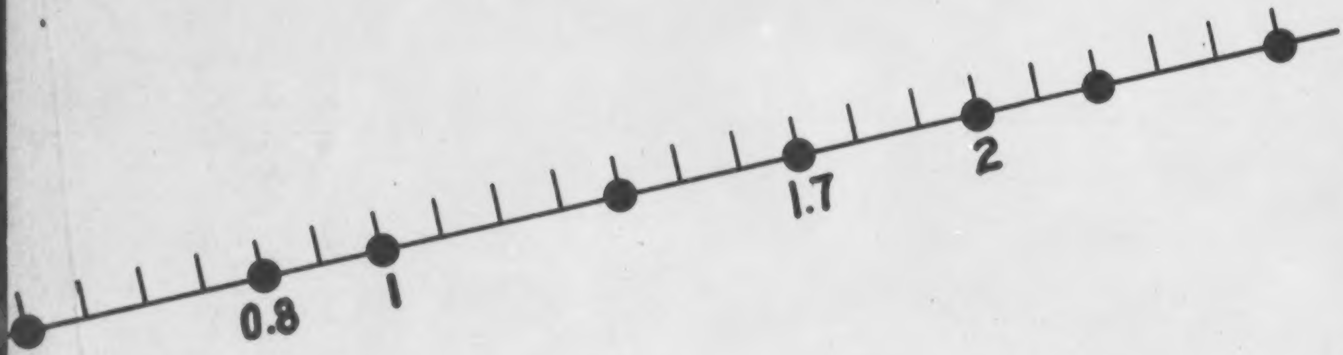
= ________

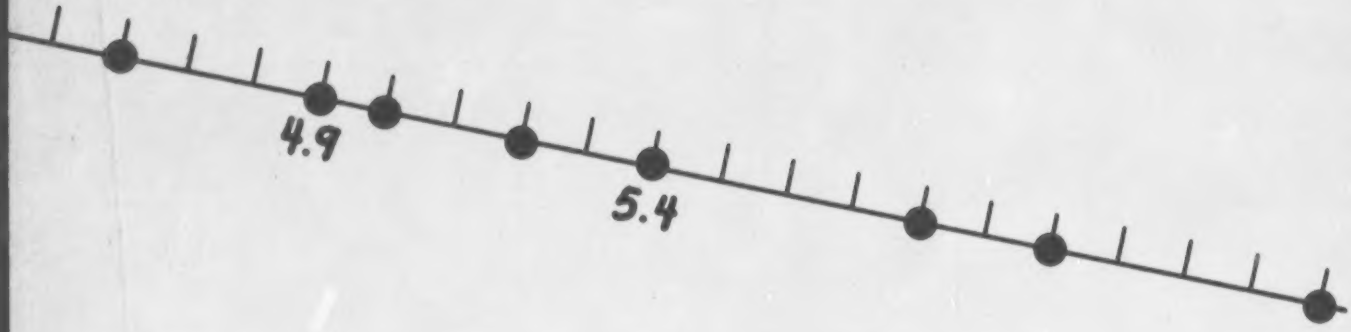
Zip is ____________.

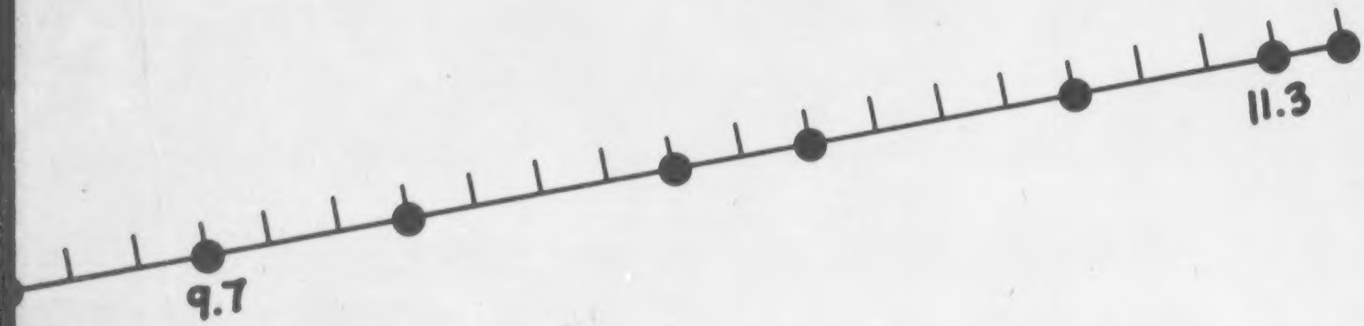
N 36 *

Name ______________________________

Label the dots.







N 37 **

Name ______________________

1.5
+ 2.7
______

10.8
+ 4.9
______

236.7
+ 95.8
______

56.7
x 2
______

1.5 + 0.7 = ______

5.8 + 0.9 = ______

0.7 + 0.8 = ______

2 x 0.7 = ______

Name ____________________

N 38 ***

$10 \times \square = 45$

$\square \div 2 = 0.8$

$2 - \square = 0.4$

$1.6 + \square = 2.5$

$\square \div 3 = 0.7$

$\square \div 10 = 3.2$

Write at least six names for the number 0.3. One is done for you.

1 - 0.7

____

____

____

____

____

____

____

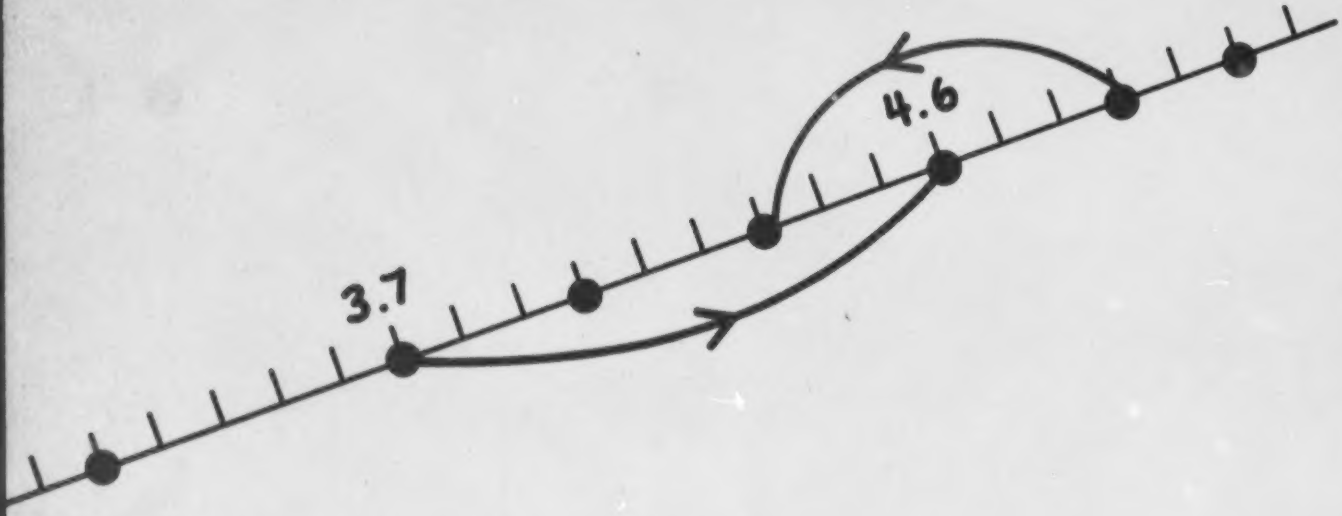
____

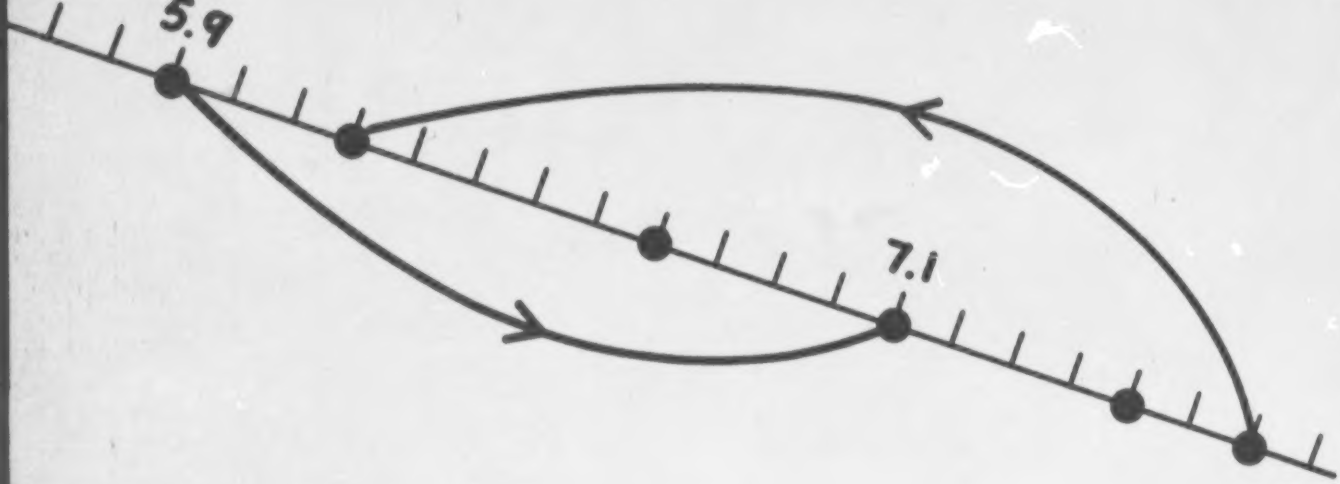
____

N 39 *

Name ______________________________

Label the dots and the arrows in each picture.



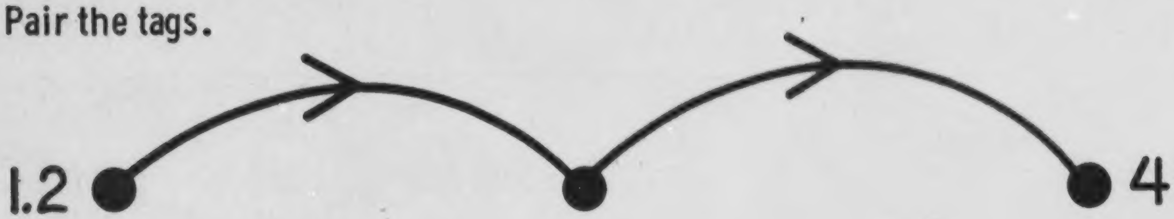


Use green to draw two +0.5 arrows.

N 40 **

Name ______________________________

Pair the tags.



+1.3

÷2

4x

−0.7

+6.8

10x

−0.8

8x

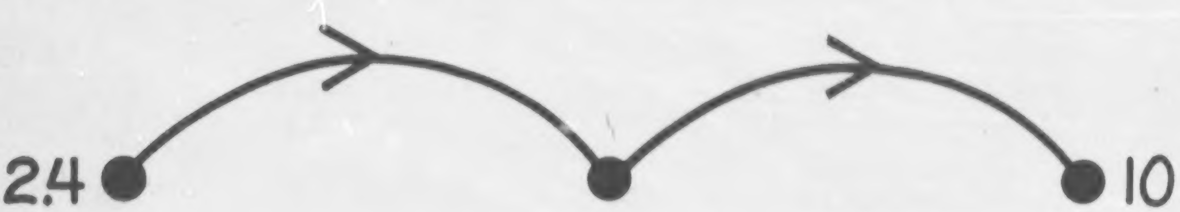
÷2

+1.5

÷3

+3.4

Name ______________________________ **N 41 *****



Complete the chart.

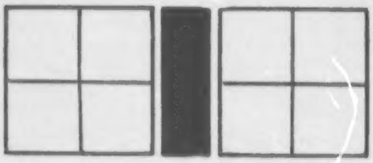
→	→
2 x	
÷ 2	
3 x	
÷ 3	
−0.9	
−1.4	
+2.6	

→	→
	2 x
	÷ 2
	10 x
	÷ 10
	− 7.4
	+ 7.4

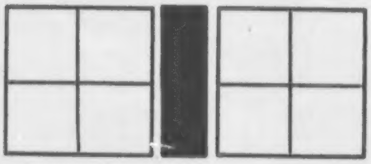
Name ____________________

N 42 *

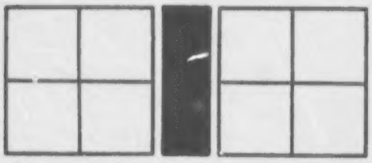
Write any number you wish on the Minicomputer using exactly four positive checkers.



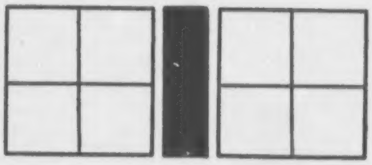
= ___



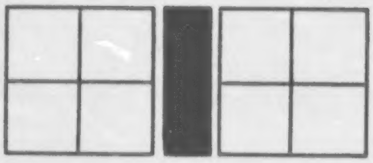
= ___



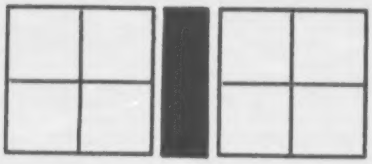
= ___



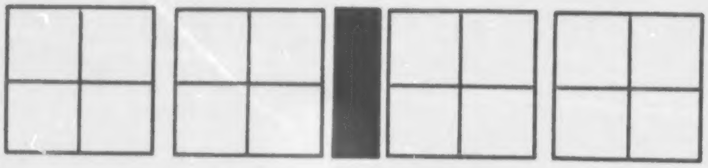
= ___



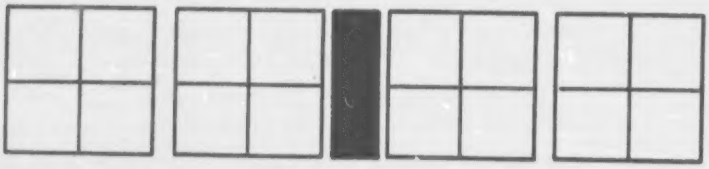
= ___



= ___



= ___

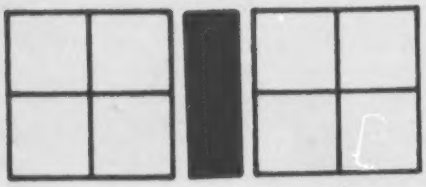


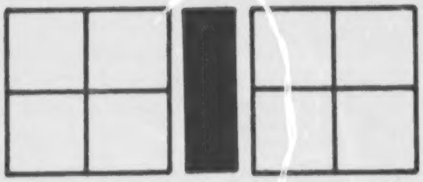
= ___

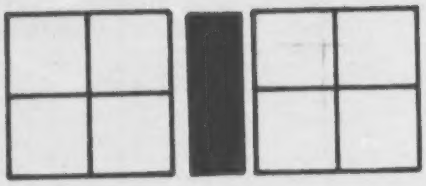
N 43 **

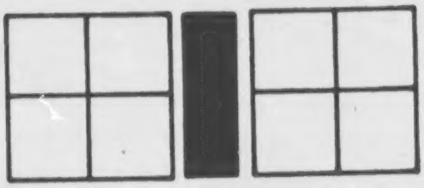
Name ______________________

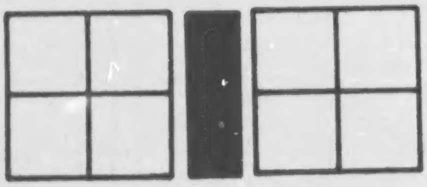
Write each of these numbers on the Minicomputer using exactly three checkers (positive or negative).

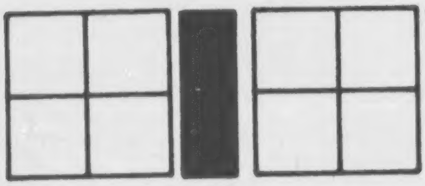

= 8.4

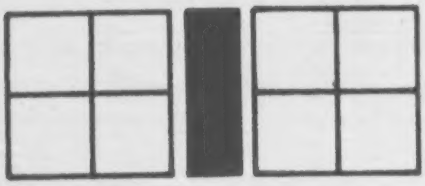

= 2

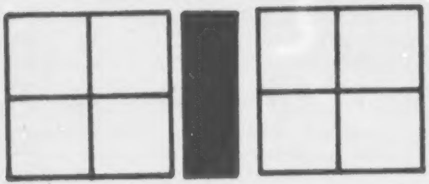

= 0.5


= 1.7


= 3.6


= 11.8


= 15

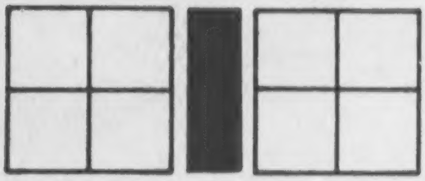

= 15.2

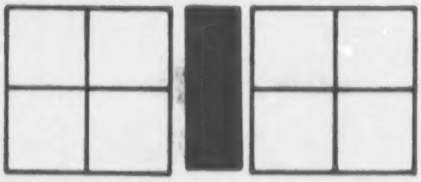
Name ______________________________

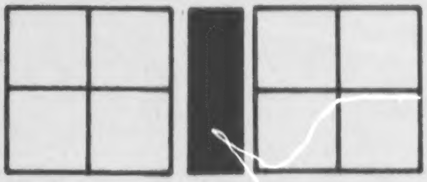
N 44 ***

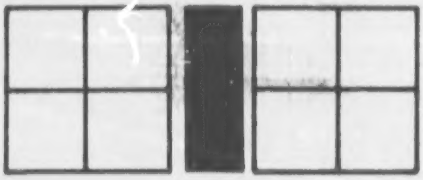
Write each of these numbers on the Minicomputer using exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨

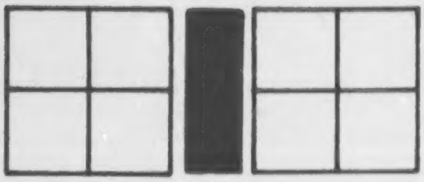

= 1.2

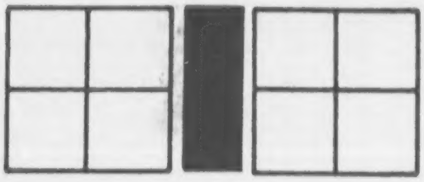

= 4.8

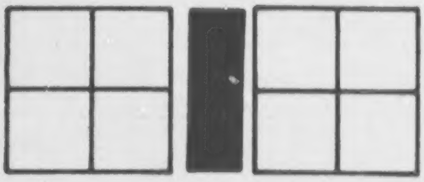

= 3.6

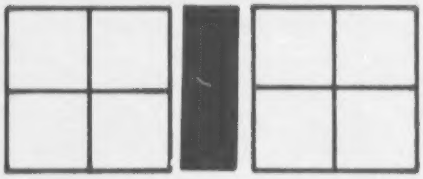

= 7.2

For the following numbers, use exactly two of the checkers above.


= 14.4


= 11.8

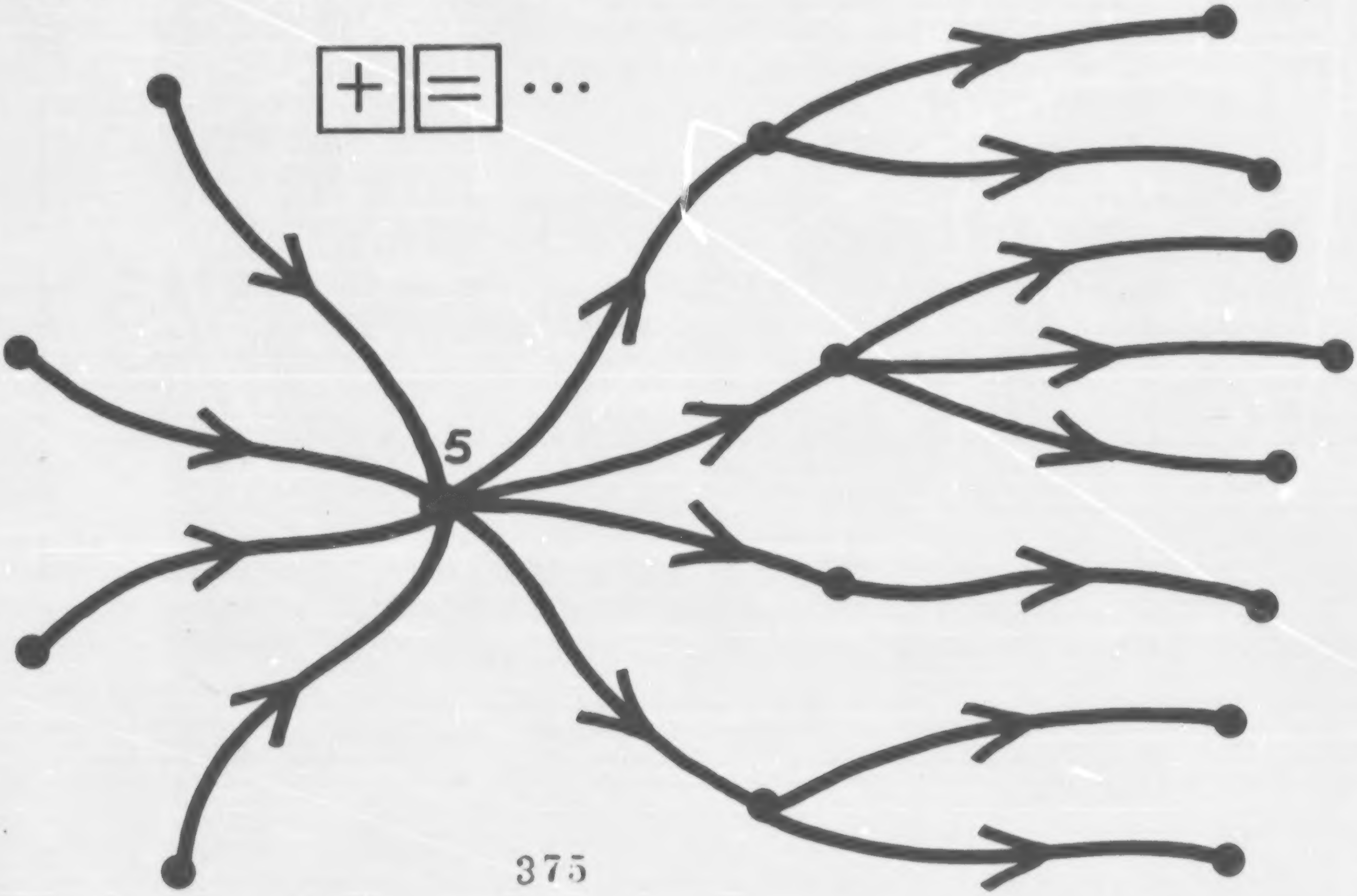

= 11.6


= 11.2

N 45 *

Name ______________________________

Label the dots. Many answers are possible. Many arrows could be added to this picture without adding any new dots. Draw at least three of them.



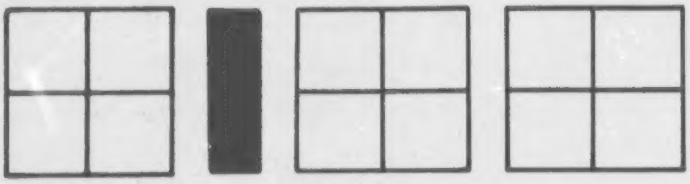
Name ______________________

N 46 **

First clue

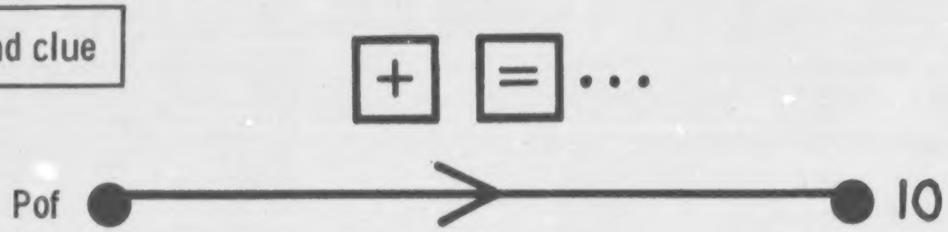
Pof is a secret number.

Pof can be put on this Minicomputer using exactly one positive checker.



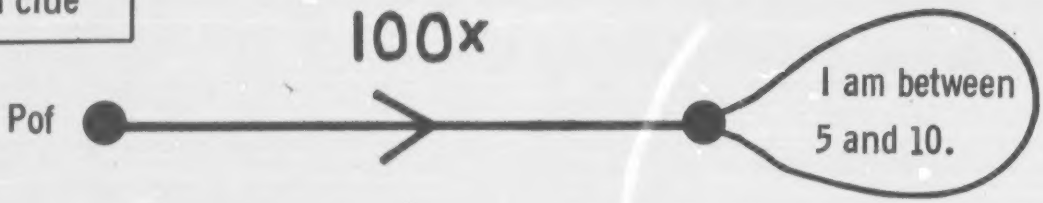
Pof is ______ or ______ or ______ or ______ or ______ or ______ or ______ or ______ or ______ or ______ or ______ or ______.

Second clue



Pof is ______ or ______ or ______ or ______ or ______ or ______ or ______ or ______ or ______.

Third clue

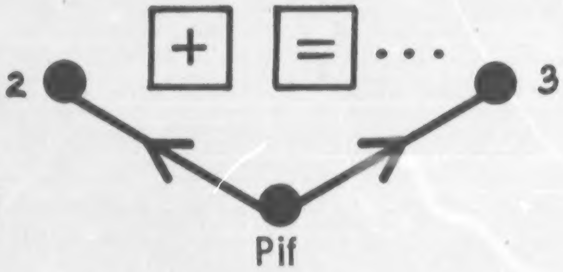

100x
Pof
I am between 5 and 10.

Pof is ______.

Name ______________________________

N 47 ***

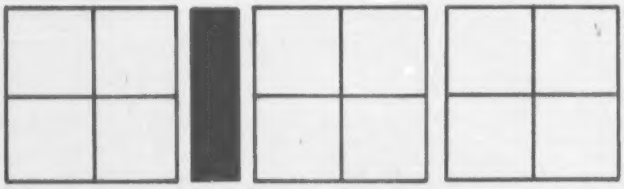
First clue



Pif could be _____; _____; _____; _____; _____; _____; _____; and so on.

Second clue

Pif can be put on this Minicomputer.



Pif is _____ or _____ or _____ or _____ or _____ or _____ or _____ or _____ or _____.

Third clue

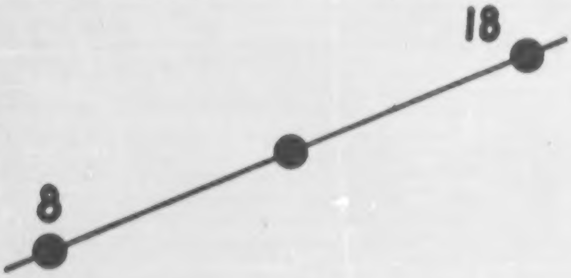
Pif cannot be put on the Minicomputer using exactly two checkers (positive or negative).

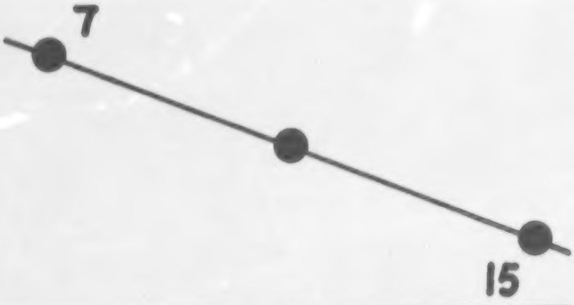
Pif is _____.

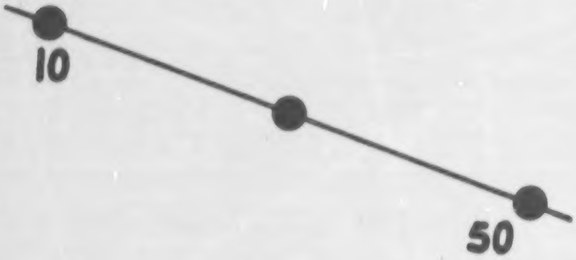
N 48 *

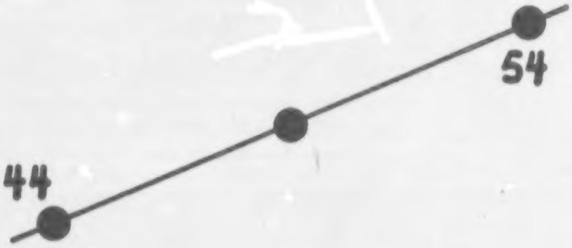
Name ______________________________

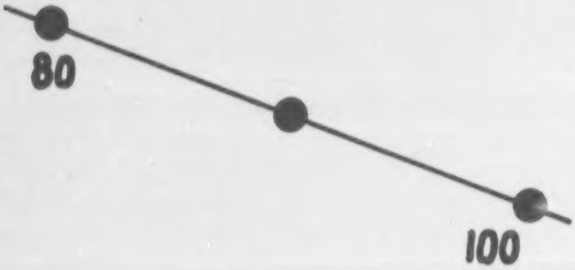
Label each midpoint.

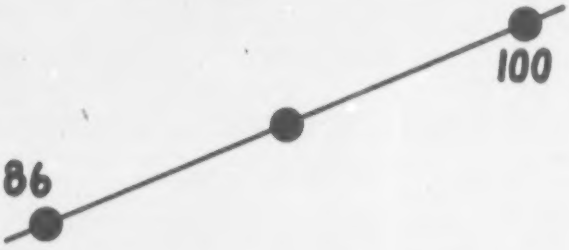


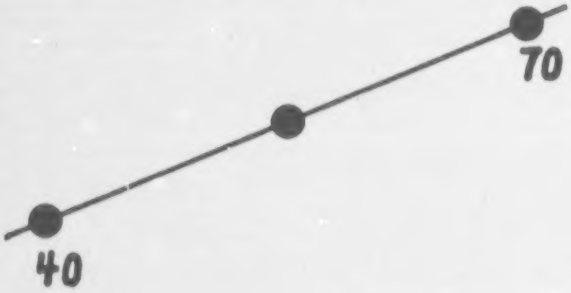


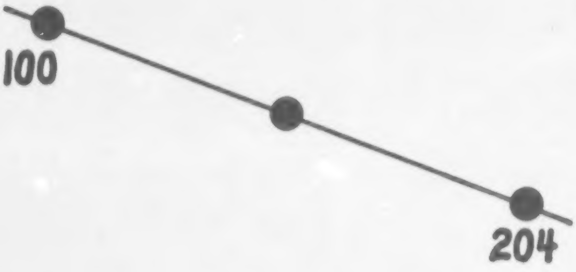








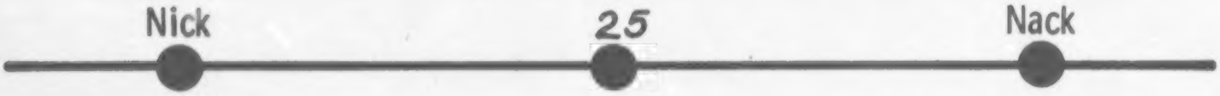




Name ______________________

N 49 **

25 is the midpoint of Nick and Nack.



Complete:

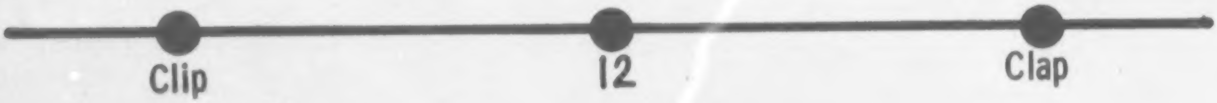
Nick	Nack
23	27
	29
20	
	32
10	
	41
0	
	49
⁻1	

Nick	Nack
19	
	33
15	
	37
5	
	47
⁻5	
24.5	
	26.5

Name ______________________________ **N 50 *****

First clue

12 is the midpoint of Clip and Clap.



If Clip were 3, Clap would be ____________.

If Clip were $^{-}2$, Clap would be ____________.

If Clap were 20, Clip would be ____________.

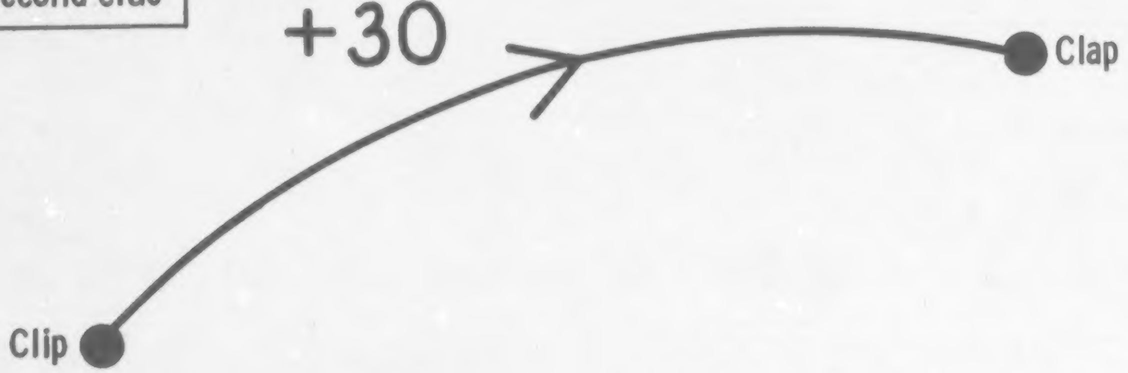
If Clap were 30, Clip would be ____________.

If Clip were $^{-}10$, Clap would be ____________.

If Clip were 9.5, Clap would be ____________.

If Clap were 15.3, Clip would be ____________.

Second clue



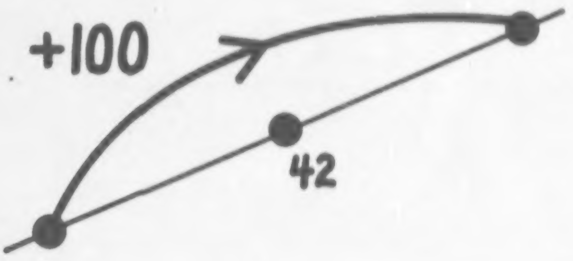
Clip is ________. Clap is ________.

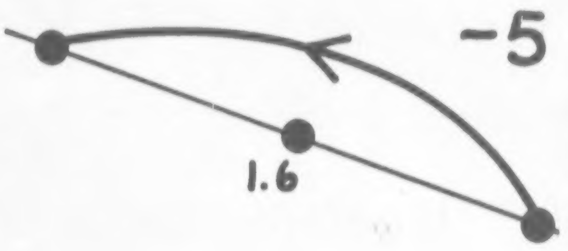
N 51 ****

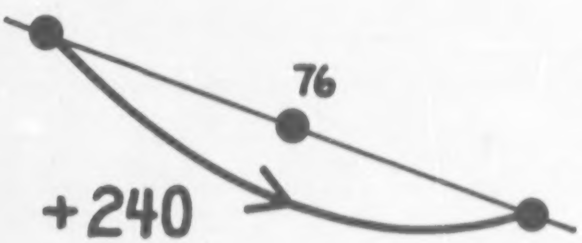
Name ______________________________

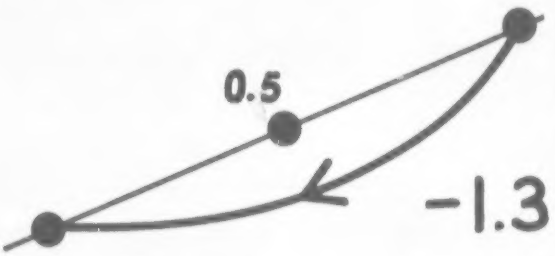
In each picture the in-between dot is the midpoint.

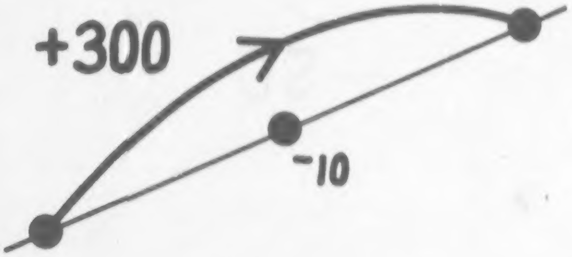
Label the dots.

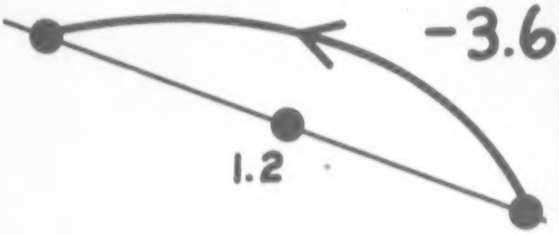


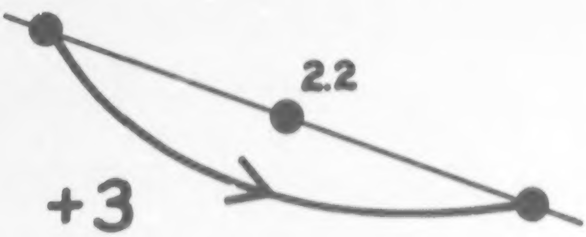


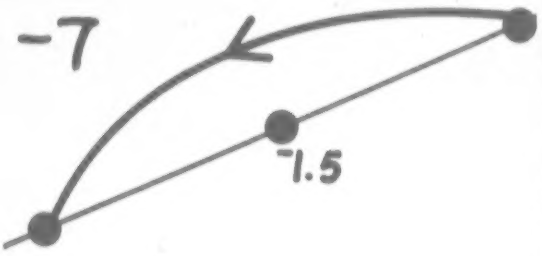










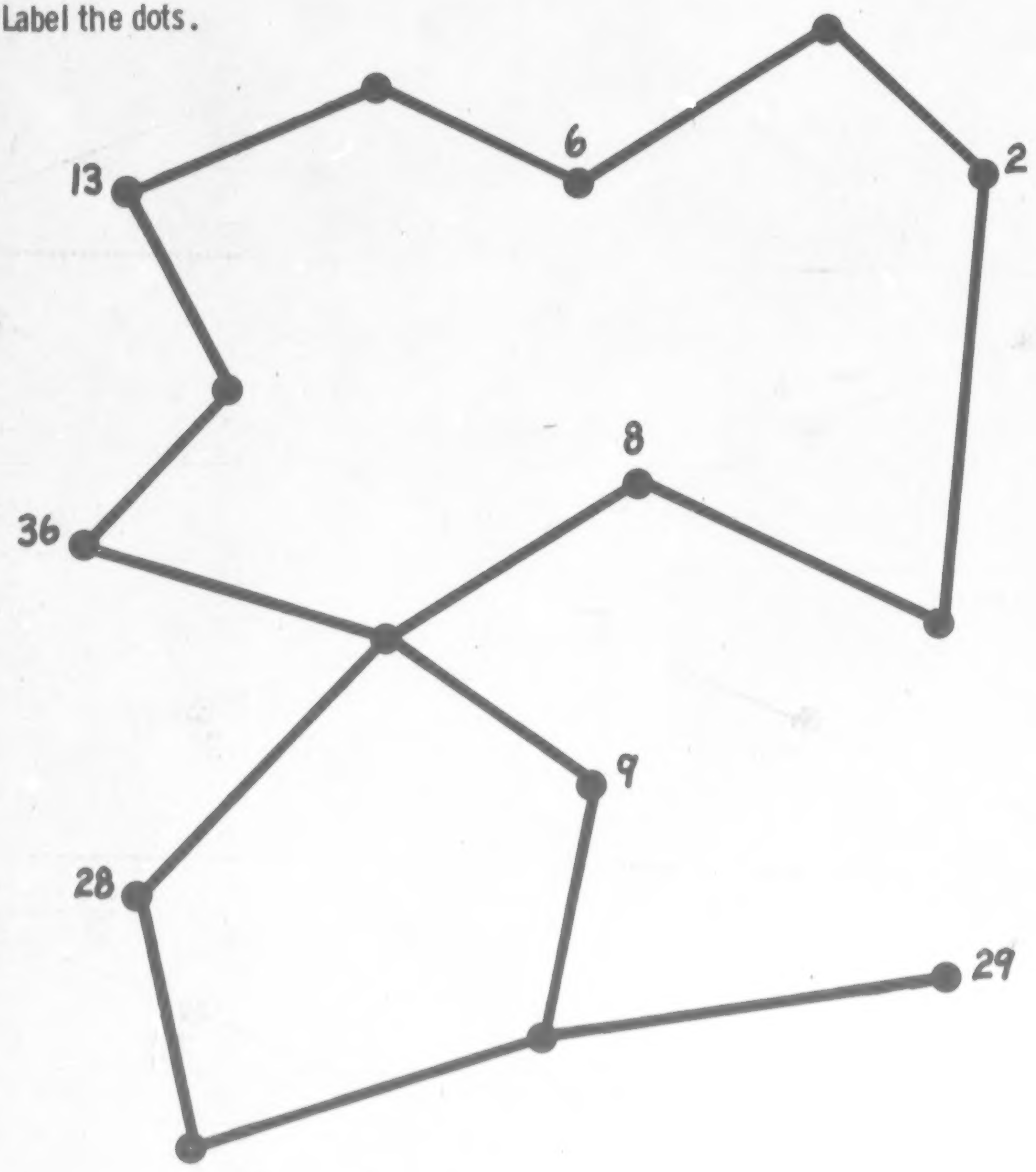


N 52 *

Name ______________________________

Telephone Game

Label the dots.

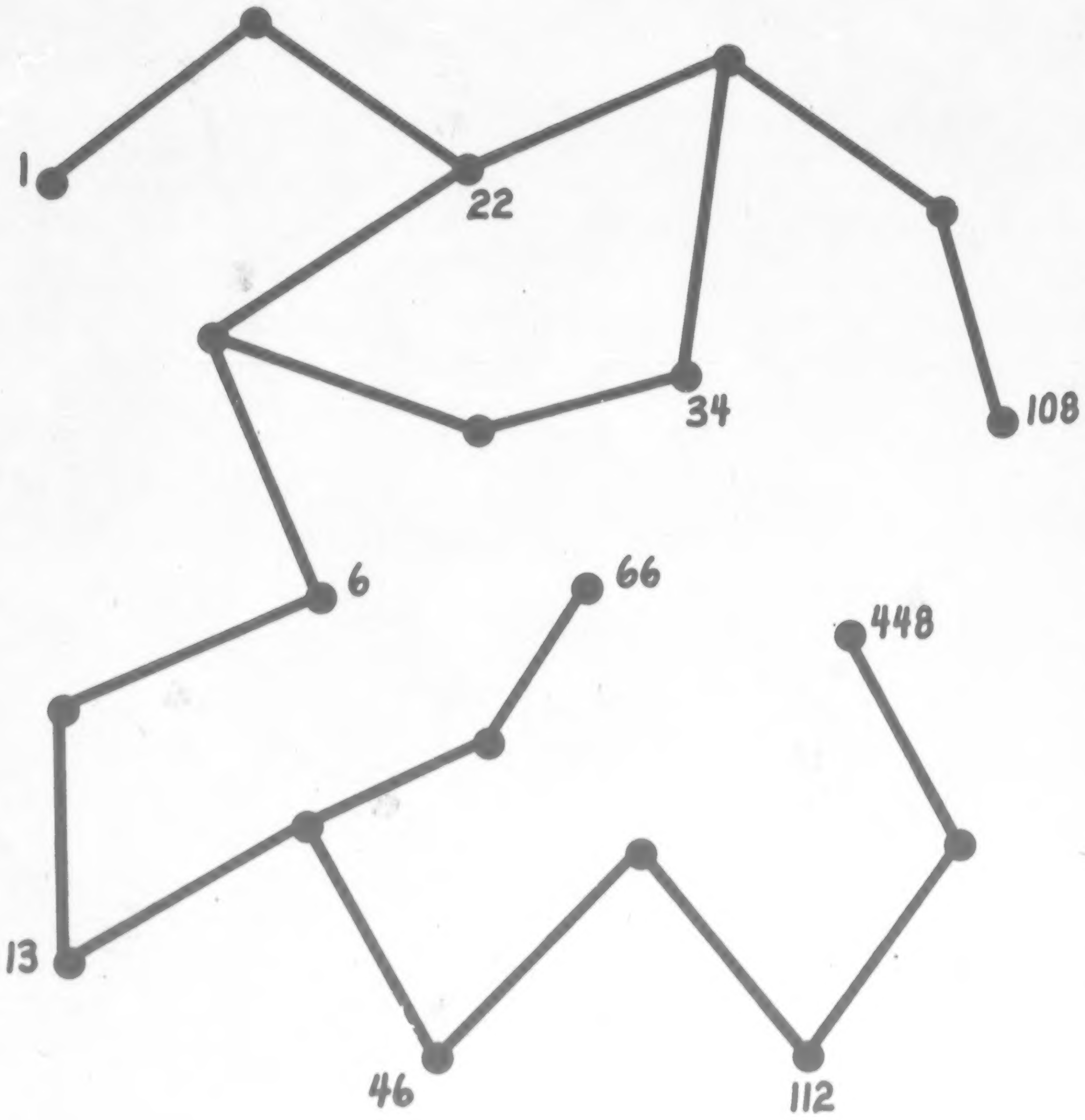

6
2
13
8
36
9
28
29

N 53 **

Name ____________________

Telephone Game

Label the dots.

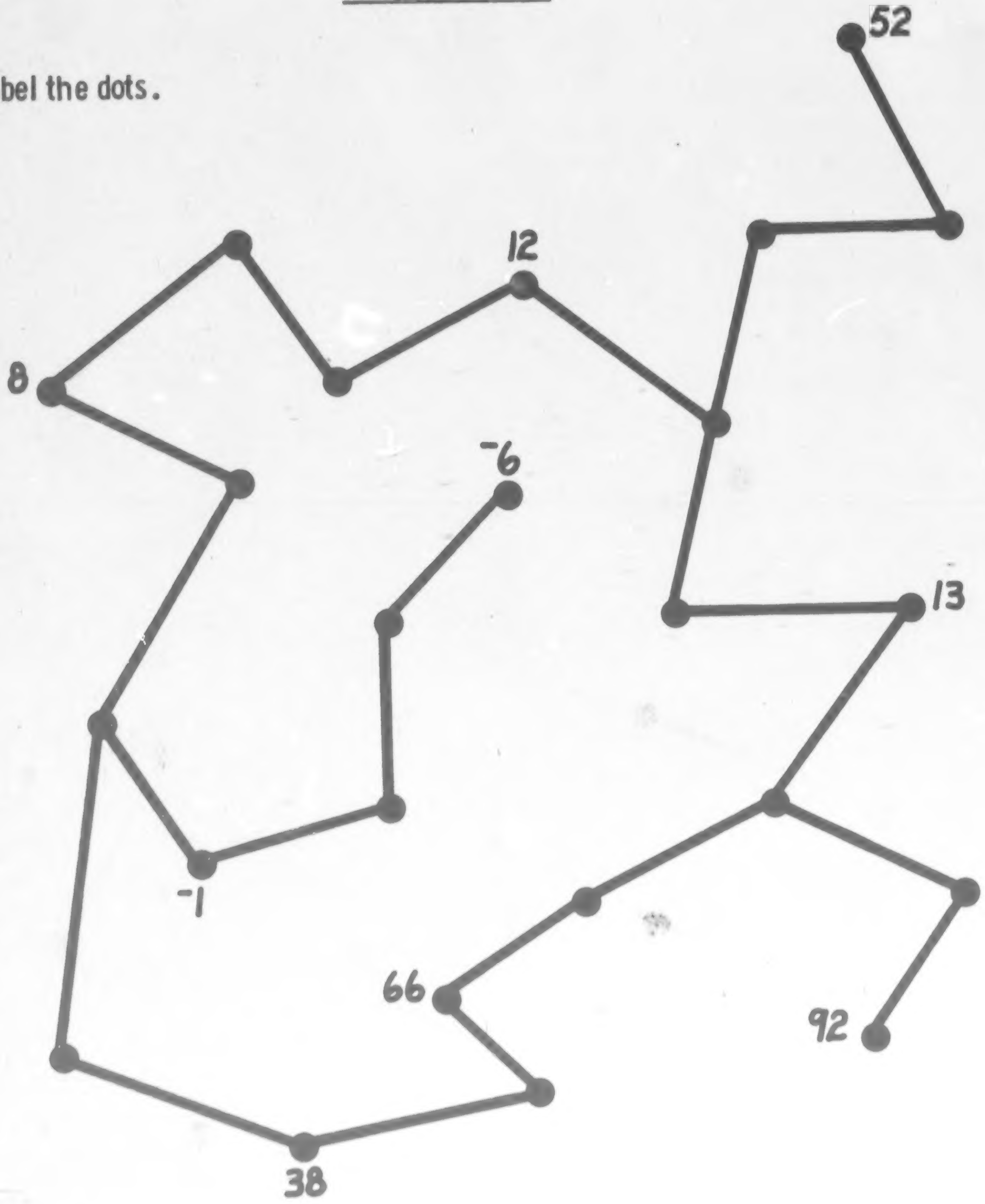

1
22
34
108
6
66
448
13
46
112

Name ______________________

N 54 ***

Telephone Game

Label the dots.



52
12
8
⁻6
13
⁻1
66
92
38

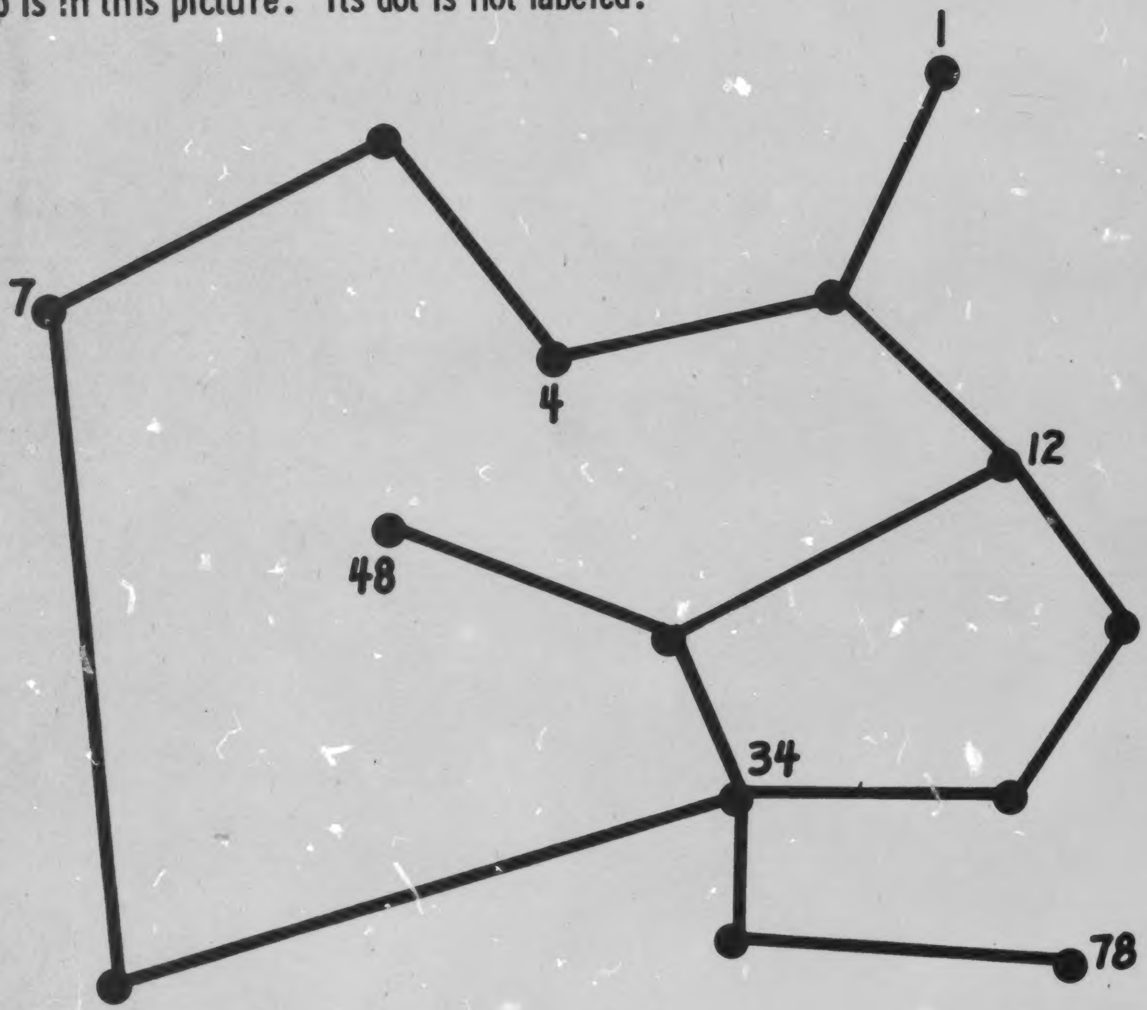
Name ______________________________

N 55 *

Telephone Game

First clue

Zip is in this picture. Its dot is not labeled.


1
7
4
12
48
34
78

Zip could be ______; ______; ______; ______; ______; ______; ______.

(over)

N 56 *

Second clue

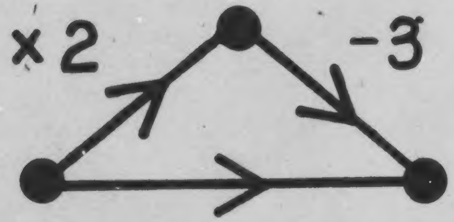
Zip can call 8 for 20¢.

Zip is _____ or _____.

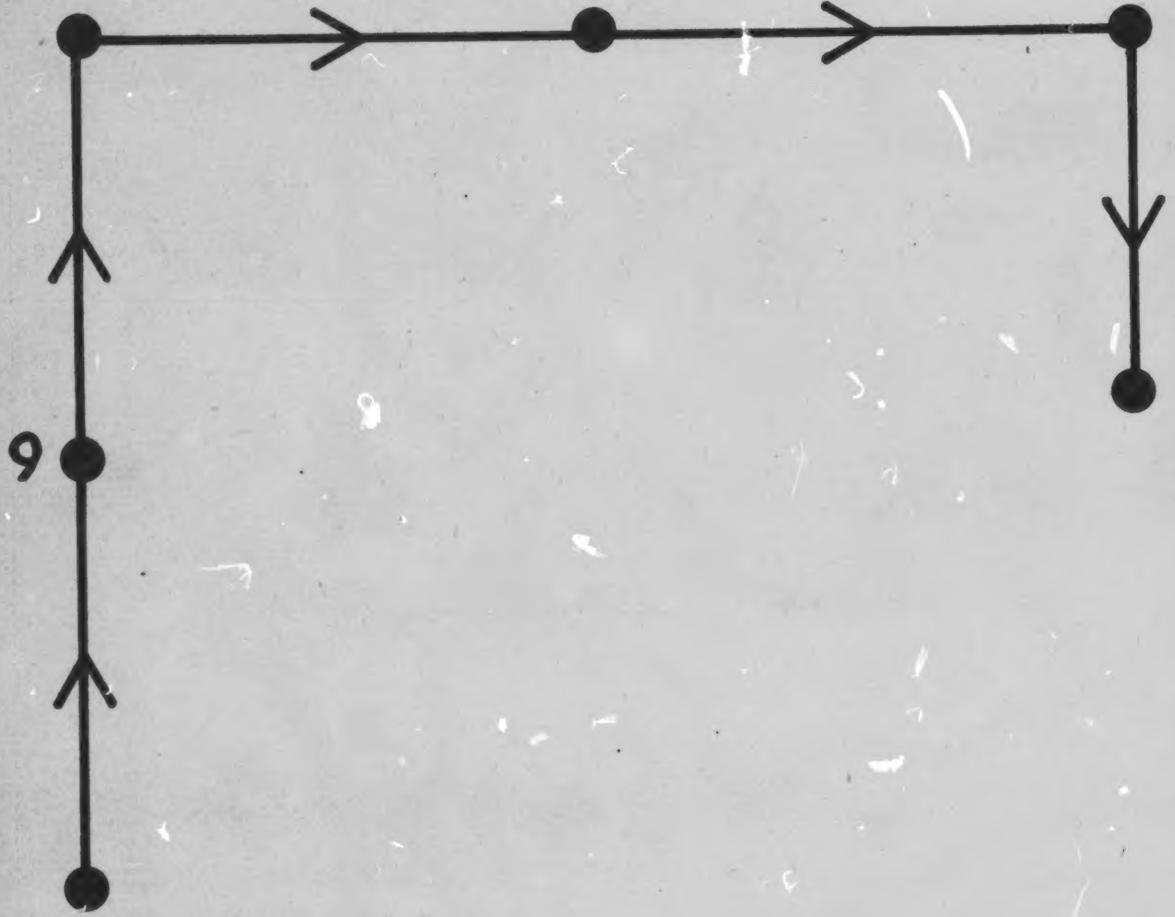
Name ______________________

N 57 *

Here is the code for the red arrows.



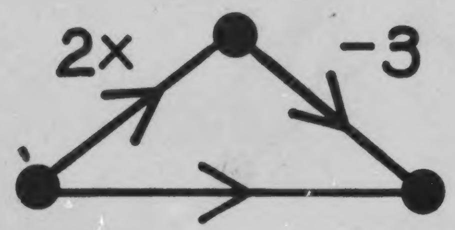
Using this code label the dots in the picture below.



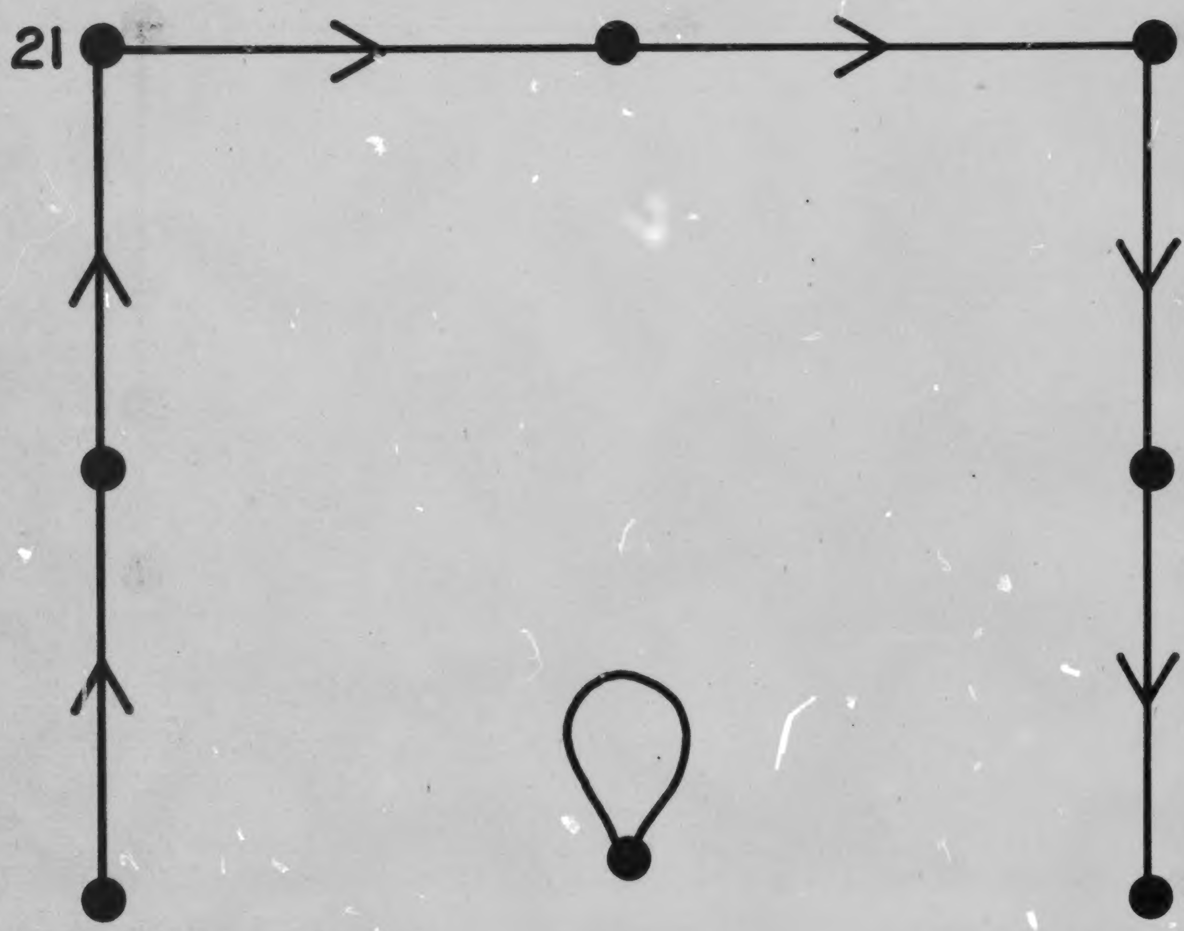
N 58 **

Name ______________________

Here is the code for the red arrows.



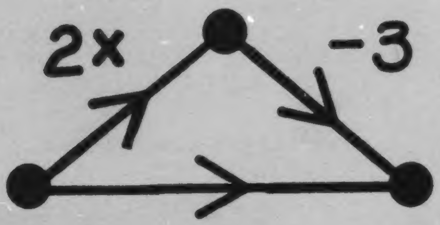
Using this code label the dots in the picture below.



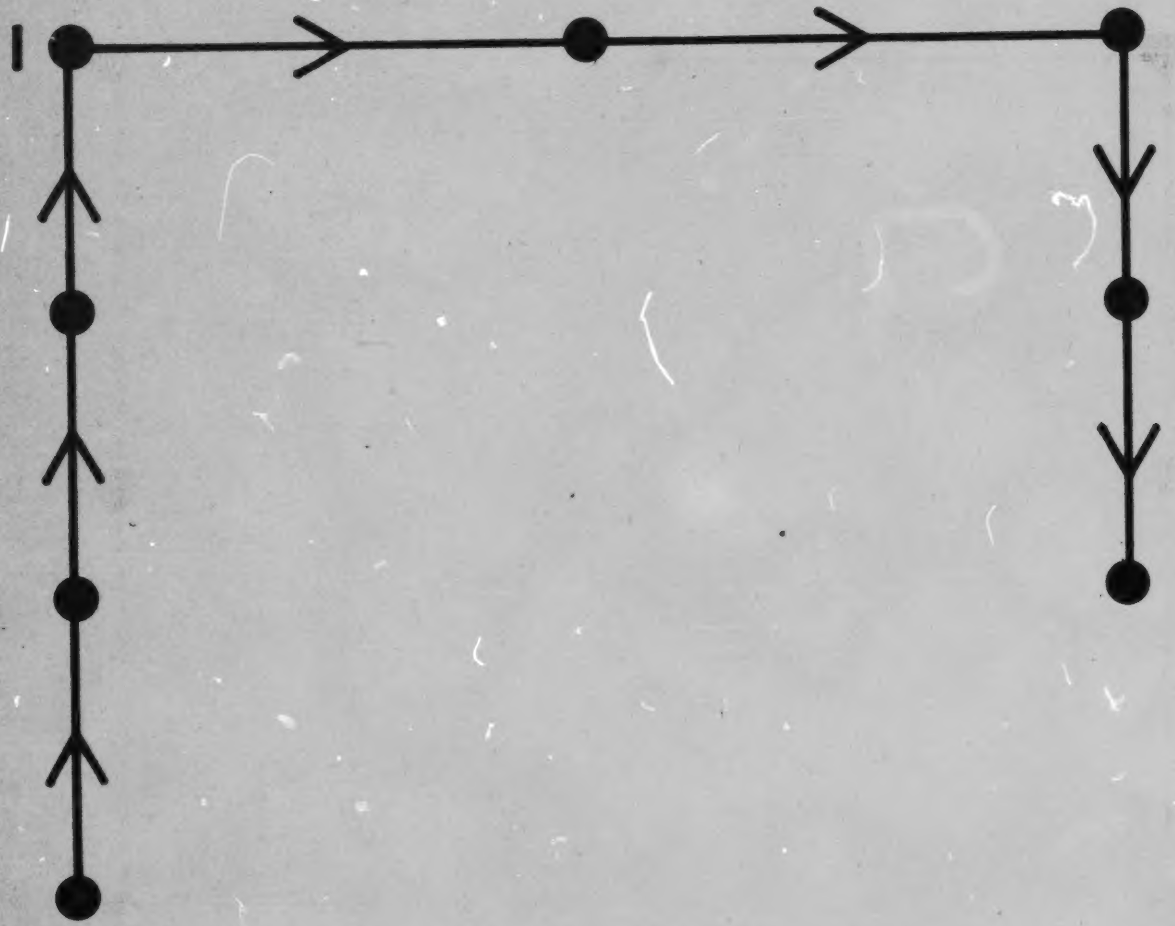
N 59 ***

Name ______________________________

Here is the code for the red arrows.



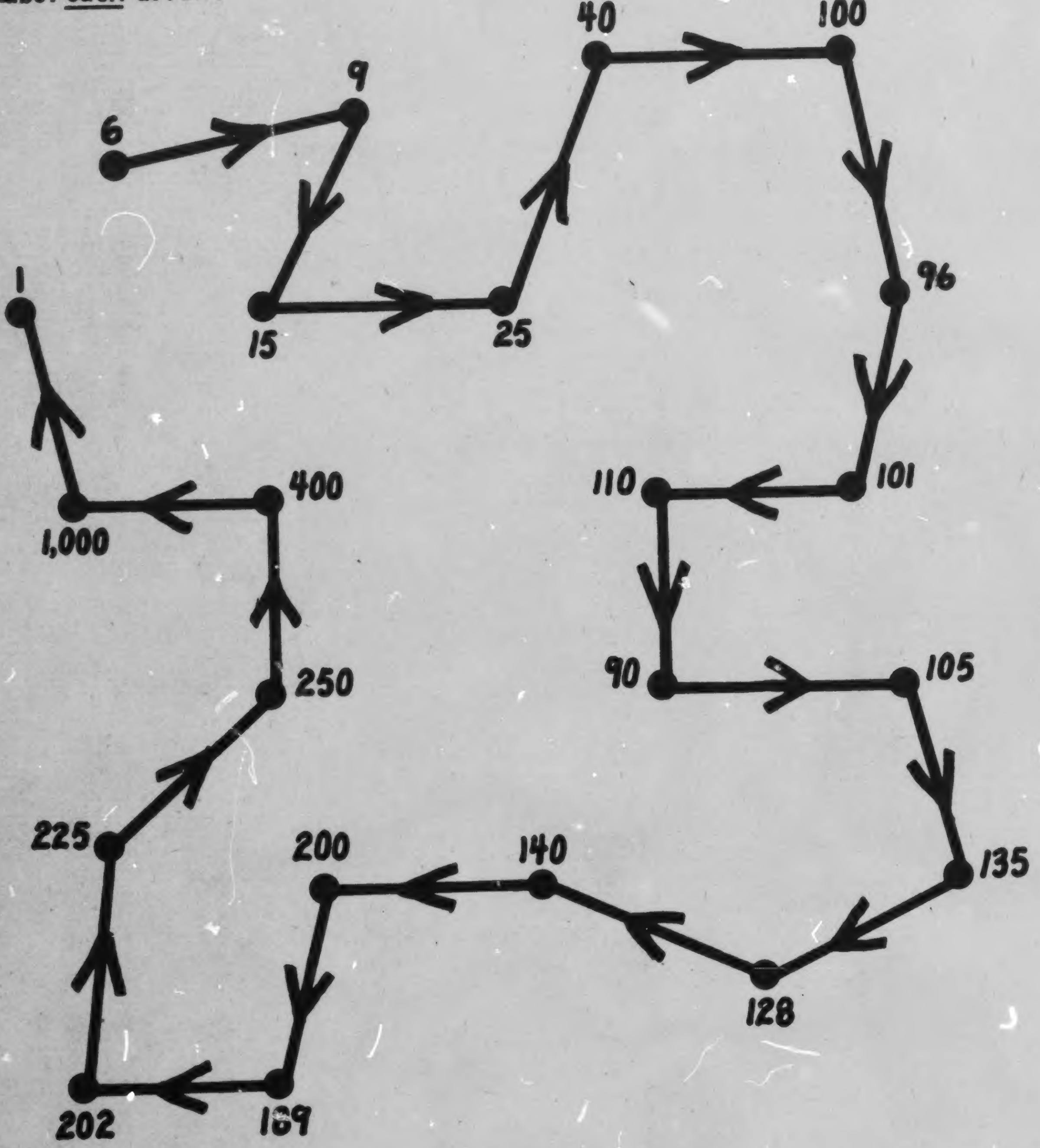
Using this code label the dots in the picture below.



Name ______________________

N 60 *

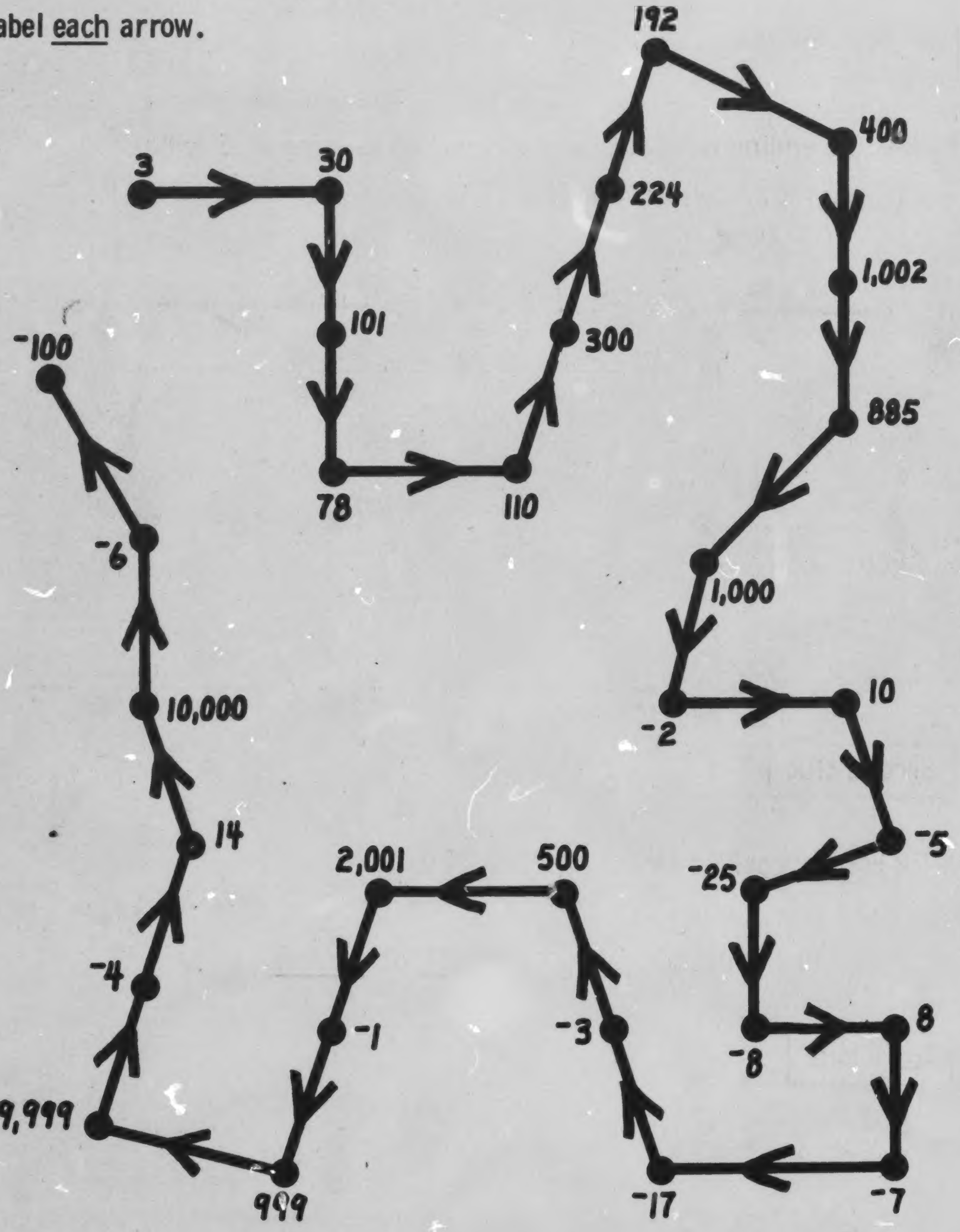
Label each arrow.


40
100
9
6
96
1
15
25
101
110
400
1,000
90
105
250
225
200
140
135
128
202
189

Name ______________________

N 61 **

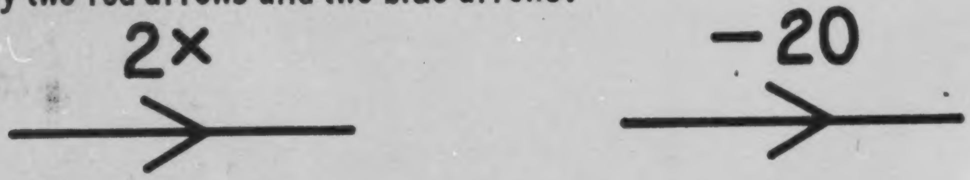
Label each arrow.


3
30
101
78
110
300
224
192
400
1,002
885
1,000
-2
10
-5
-25
-8
8
-7
-17
-3
500
2,001
-1
999
9,999
-4
14
10,000
-6
-100

Name ______________________________ **N 62 *****

First clue

Pick is the ending number of a red-blue road starting at 35 and using exactly two red arrows and two blue arrows.



35●

Pick could be ______; ______; ______; ______; ______ or ______.

Second clue

Pick is not a positive divisor of 1,000.

Pick could be ______; ______ or ______.

Third clue

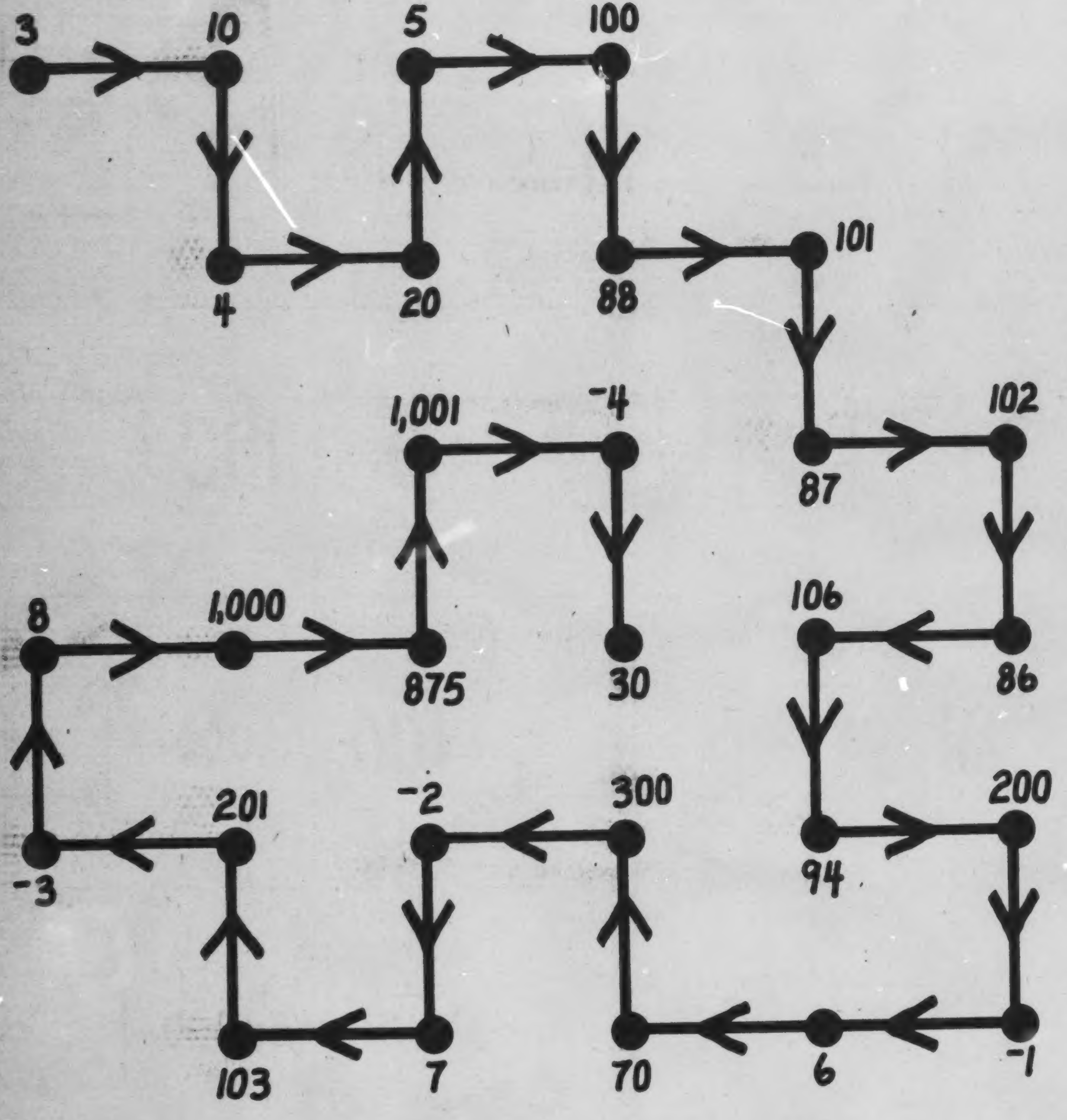
Pick is smaller than 50.

Pick is ______.

Name ____________________

N 63 *

Label each arrow.


3
10
4
20
5
100
88
101
87
102
86
106
94
200
-1
6
70
300
-2
7
103
201
-3
8
1,000
875
1,001
-4
30

Name ____________________

N 64 **

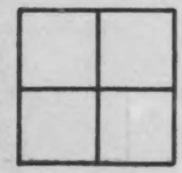
Each of these is a different name for the same number. Complete them.

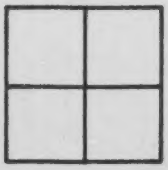
51 – 28	71 – ☐	101 – ☐
31 – ☐	☐ – 0	100 – ☐
81 – ☐	☐ – 100	1,000 – ☐
201 – ☐	☐ – 1,000	20,000 – ☐

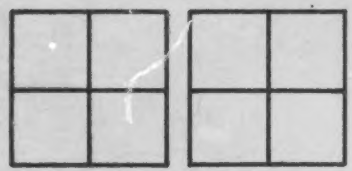
Name ______________________

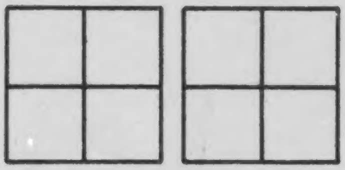
N 65 ***

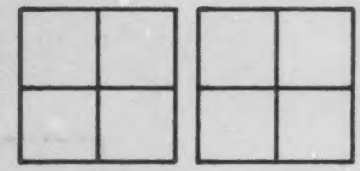
Put these numbers on the Minicomputer using exactly one positive checker and two negative checkers.

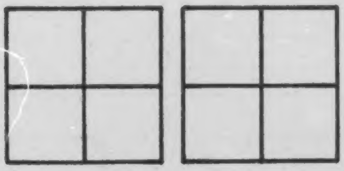

= $^{-}7$

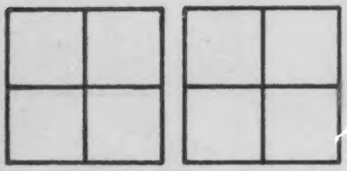

= $^{-}15$

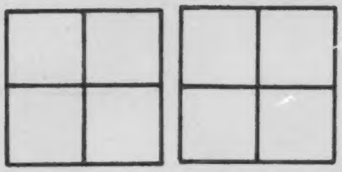

= $^{-}13$

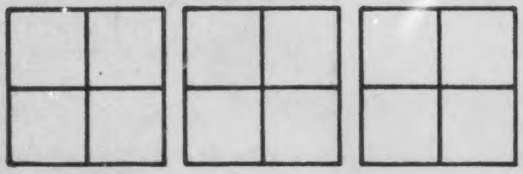

= $^{-}27$


= $^{-}36$


= $^{-}74$


= $^{-}77$


= $^{-}152$

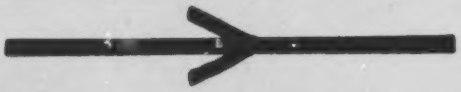

= $^{-}788$

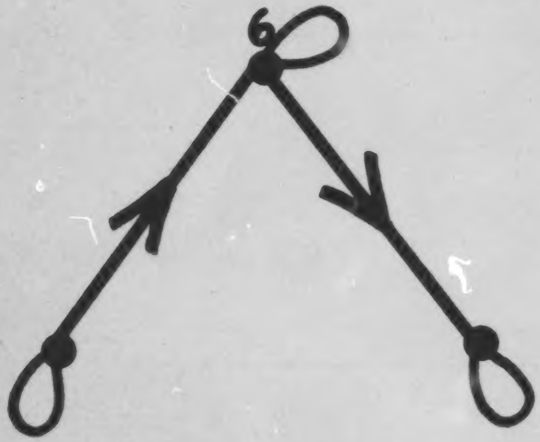
N 66 *

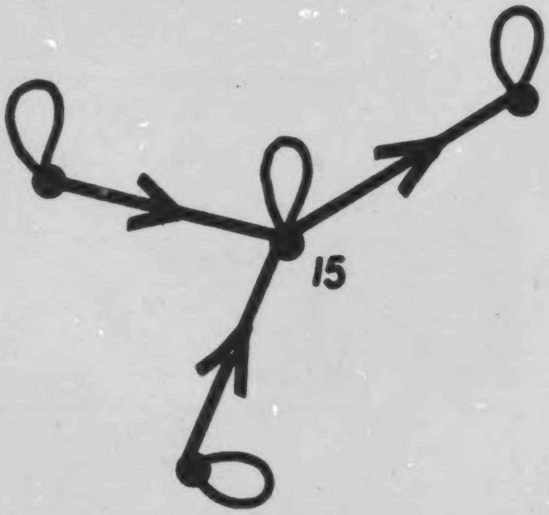
Name ______________________________

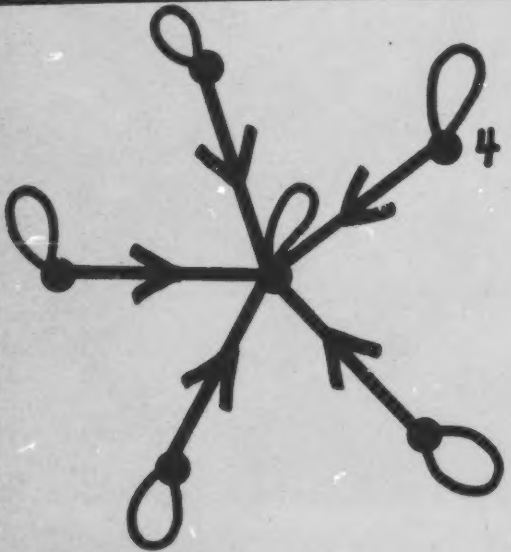
Label the dots and draw the missing arrows.

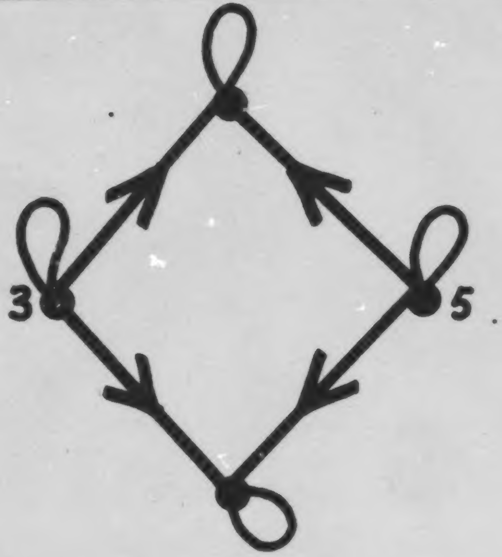
is a positive divisor of





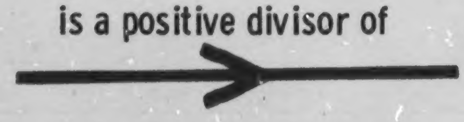


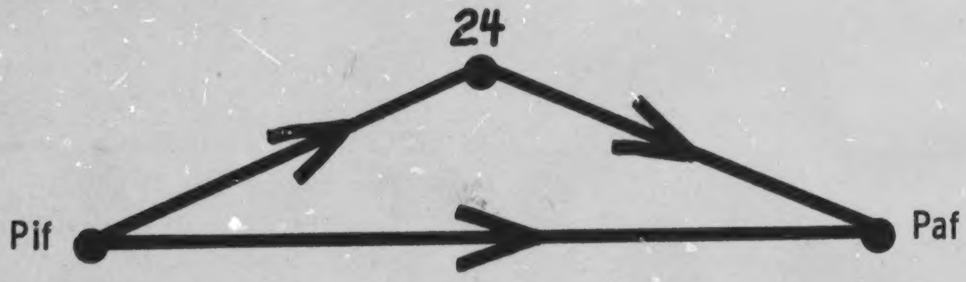




N 67 **

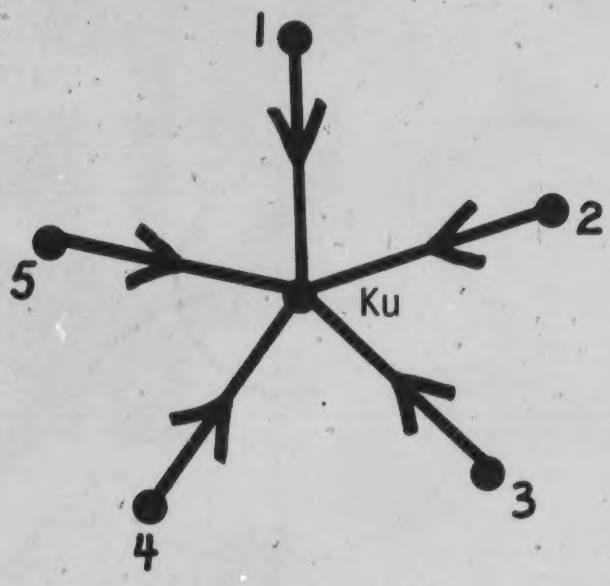
Name ______________________________


is a positive divisor of



Pif could be _____; _____; _____; _____; _____; _____; _____.

Paf could be _____; _____; _____; _____; _____; _____; and so on.



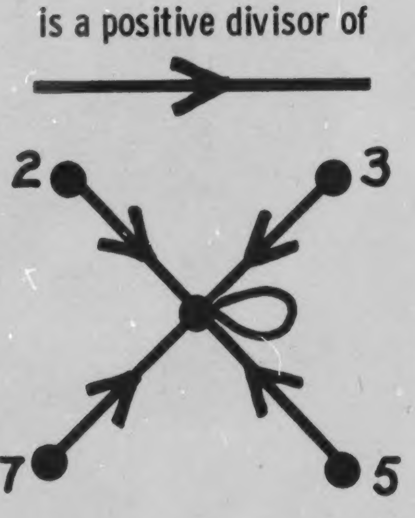
Ku could be _____; _____; _____; _____; _____; _____; and so on.

N 68 ***

Name ______________________________

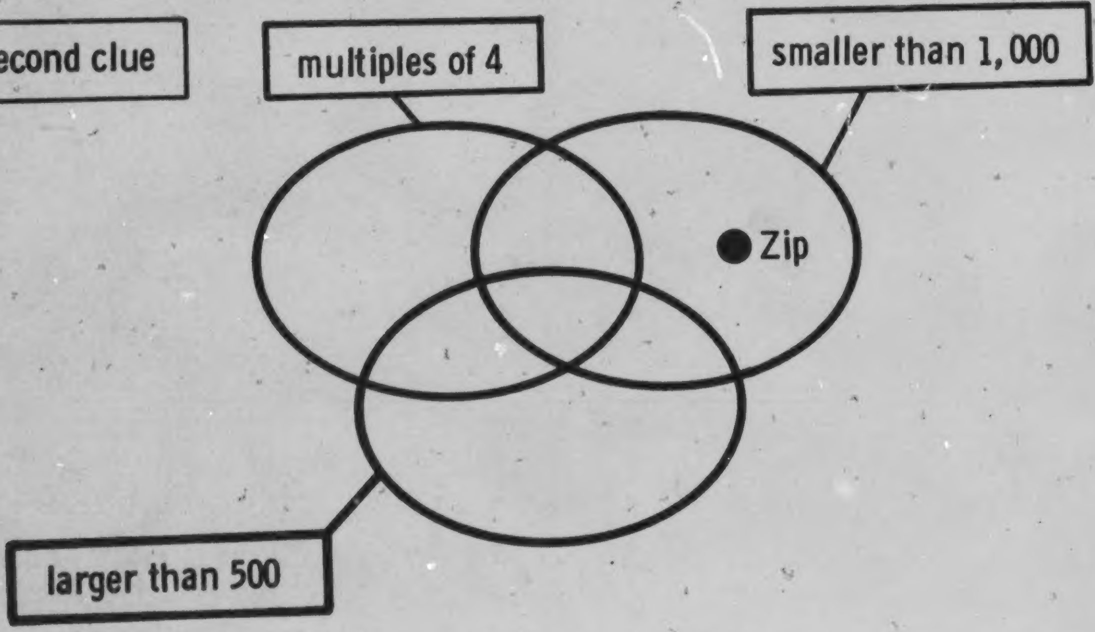
Zip is a positive integer.

First clue


is a positive divisor of
2
3
7
5

Zip could be ______; ______; ______; ______; ______; and so on.

Second clue


multiples of 4
smaller than 1,000
Zip
larger than 500

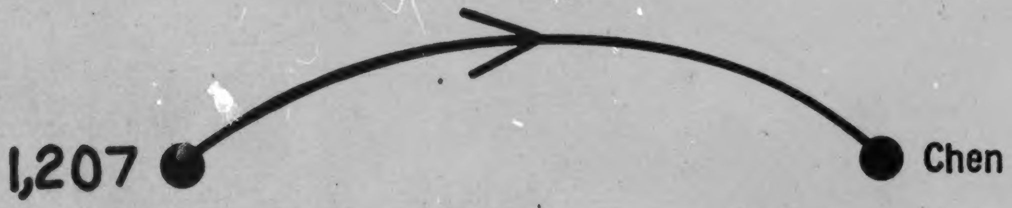
Zip is ______.

N 69 *

Name ______________________________

Chen is a secret number.

First clue $\boxed{-}\ \boxed{5}\ \boxed{=}$ . . .

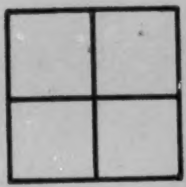


Chen could be ______; ______; ______; ______; ______; and so on.

What is the smallest positive number that Chen could be? __________

Second clue

Chen can be put on this Minicomputer with exactly three positive checkers.



Chen is _______ or _______ or _______.

Third clue

If you add 8 to Chen, you will get a square number.

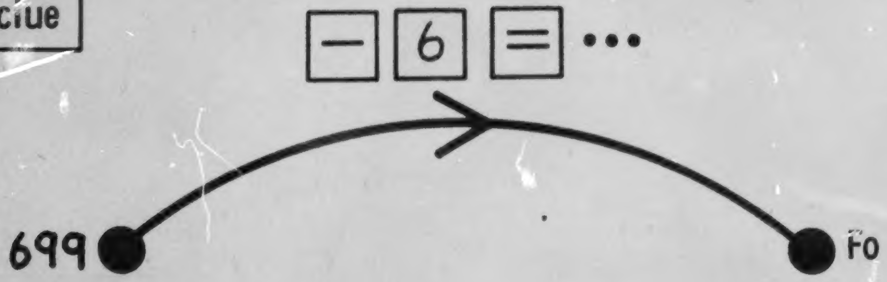
Chen is ________.

Name ______________________________

N 70 **

Fo is a secret number.

First clue



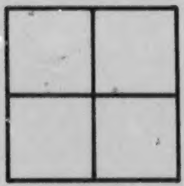
Fo could be ______; ______; ______; ______; ______; ______; and so on.

What is the smallest positive number that Fo could be ? ________

What is the largest negative number that Fo could be ? ________

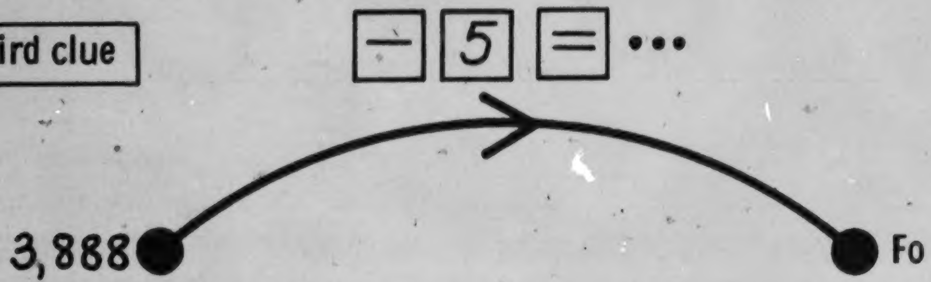
Second clue

Fo can be put on this Minicomputer with exactly three checkers (positive or negative).



Fo is ______ or ______ or ______ or ______ or ______ or ______.

Third clue

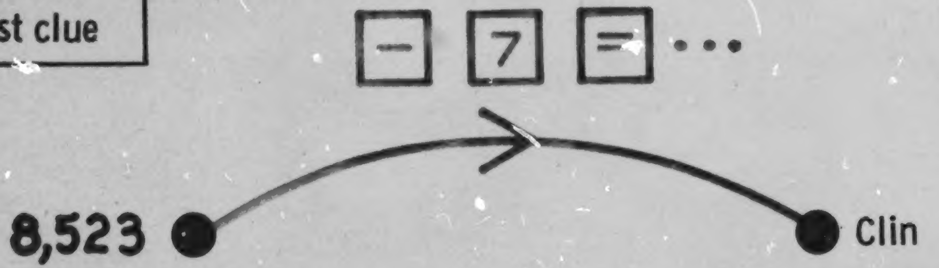


Fo is ________.

Name ______________________________

N 71 ***

First clue



What is the smallest positive number that Clin could be ? ____________

What is the largest negative number that Clin could be ? ____________

Second clue

Clin is a positive number smaller than 100.

Clin cannot be put on the Minicomputer with exactly three checkers (positive or negative).

Clin is __________ or __________.

Third clue

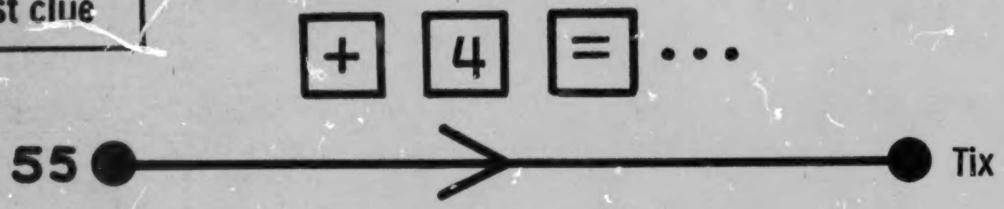
If you add 20 to Clin, you will find a prime number.

Clin is ______.

Name ______________________________ **N 72 ***

Tix is a secret number.

First clue



Tix could be _____; _____; _____; _____; _____; _____; _____; _____;

_____; _____; _____; _____; _____; and so on.

What is the smallest number that Tix could be ? __________

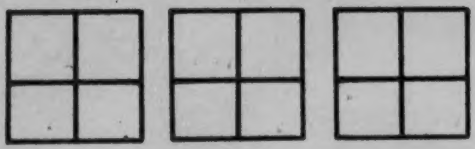
Second clue

Tix is between 190 and 220.

Tix is ____ or ____ or ____ or ____ or ____ or ____ or ____ or ____.

Third clue

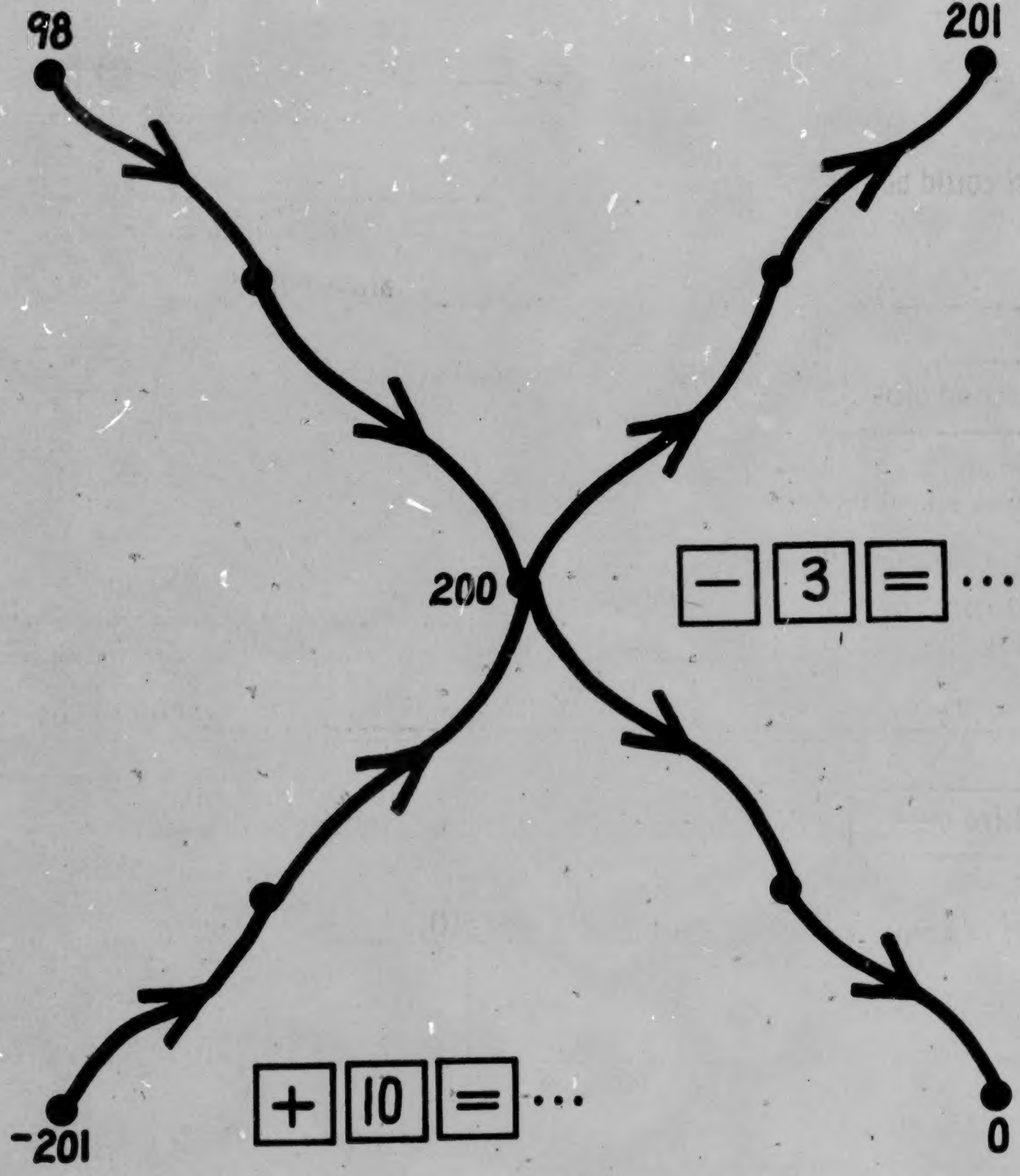
Tix can be put on this Minicomputer using exactly one positive and one negative checker.



Tix is ______.

Name ________________________ N 73 **

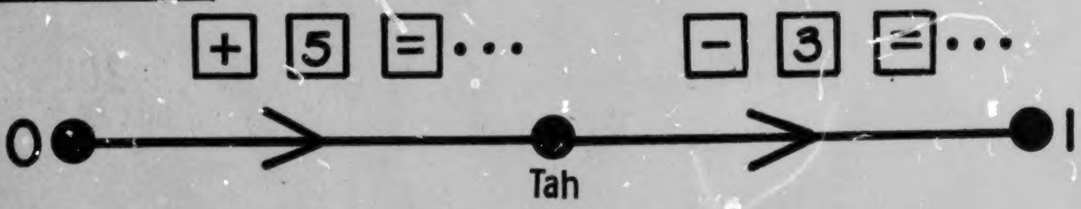
Label the dots. Many answers are possible.


98
201
200
−
3
=
…
+
10
=
…
-201
0

N 74 ***

Name ______________________________

First clue



Tah could be ____; ____; ____; ____; ____; ____; ____; ____; ____;

____; ____; ____; ____; ____; ____; ____; and so on.

Second clue

Tah is a multiple of 4.

Tah could be ______; ______; ______; ______; ______; ______; ______;

______; ______; ______; ______; ______; ______; ______; and so on.

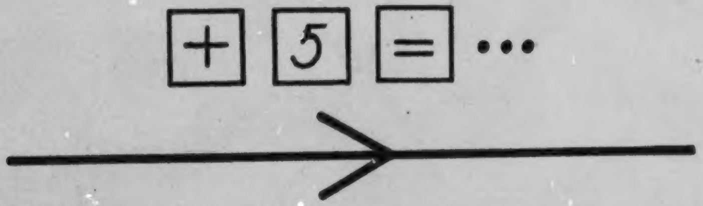
Third clue

Tah is a square number between 100 and 500.

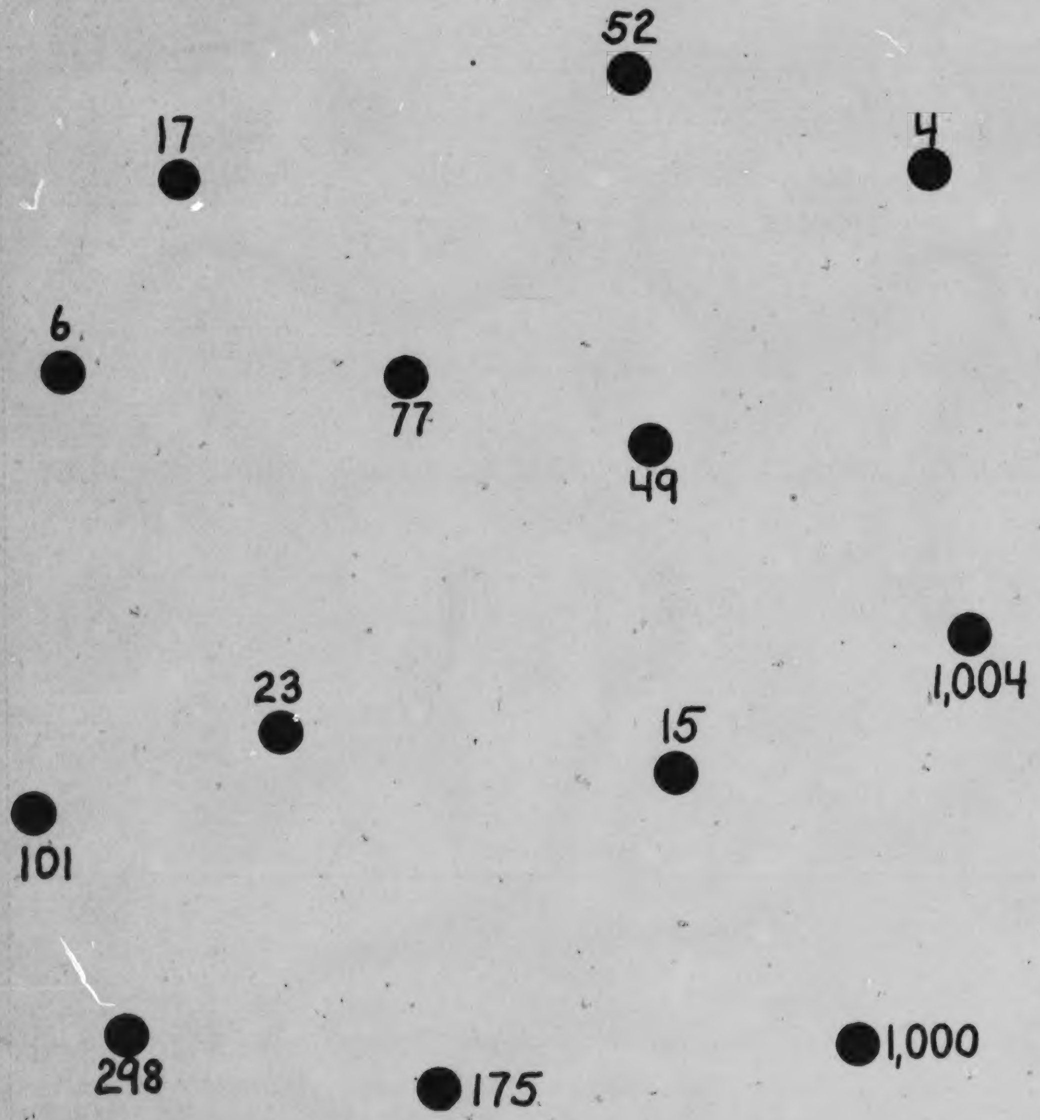
Tah is ______.

Name ____________________

N 75 *



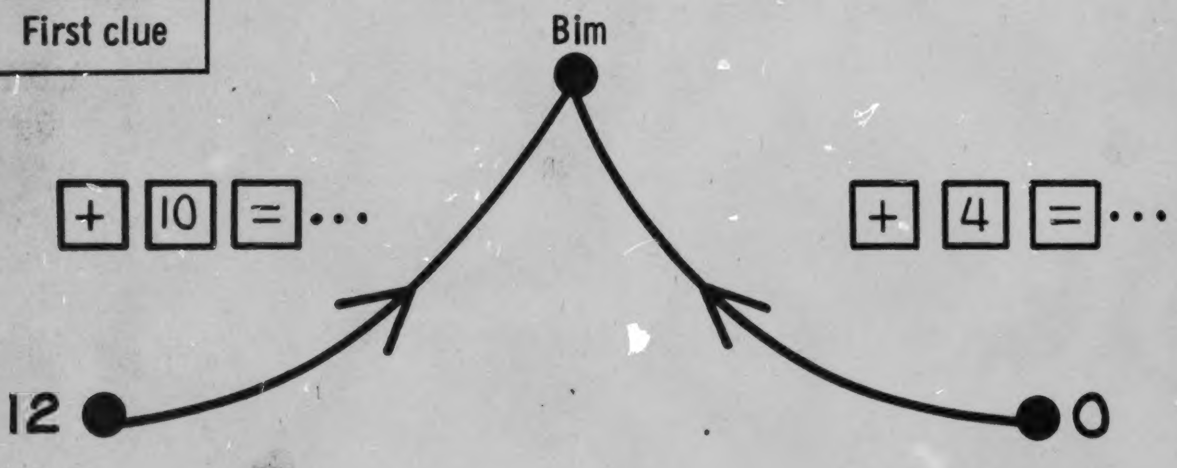
Draw as many red arrows as you can between these dots.


52
17
4
6
77
99
49
1,004
23
15
101
298
175
1,000

Name ______________________________

N 76 **

First clue



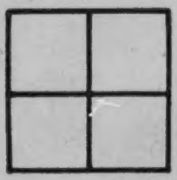
Bim could be ____; ____; ____; ____; ____; ____; ____; ____;

and so on.

Second clue

Bim can be put on this Minicomputer using exactly one of these checkers:

② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨



Bim is ________ or ________.

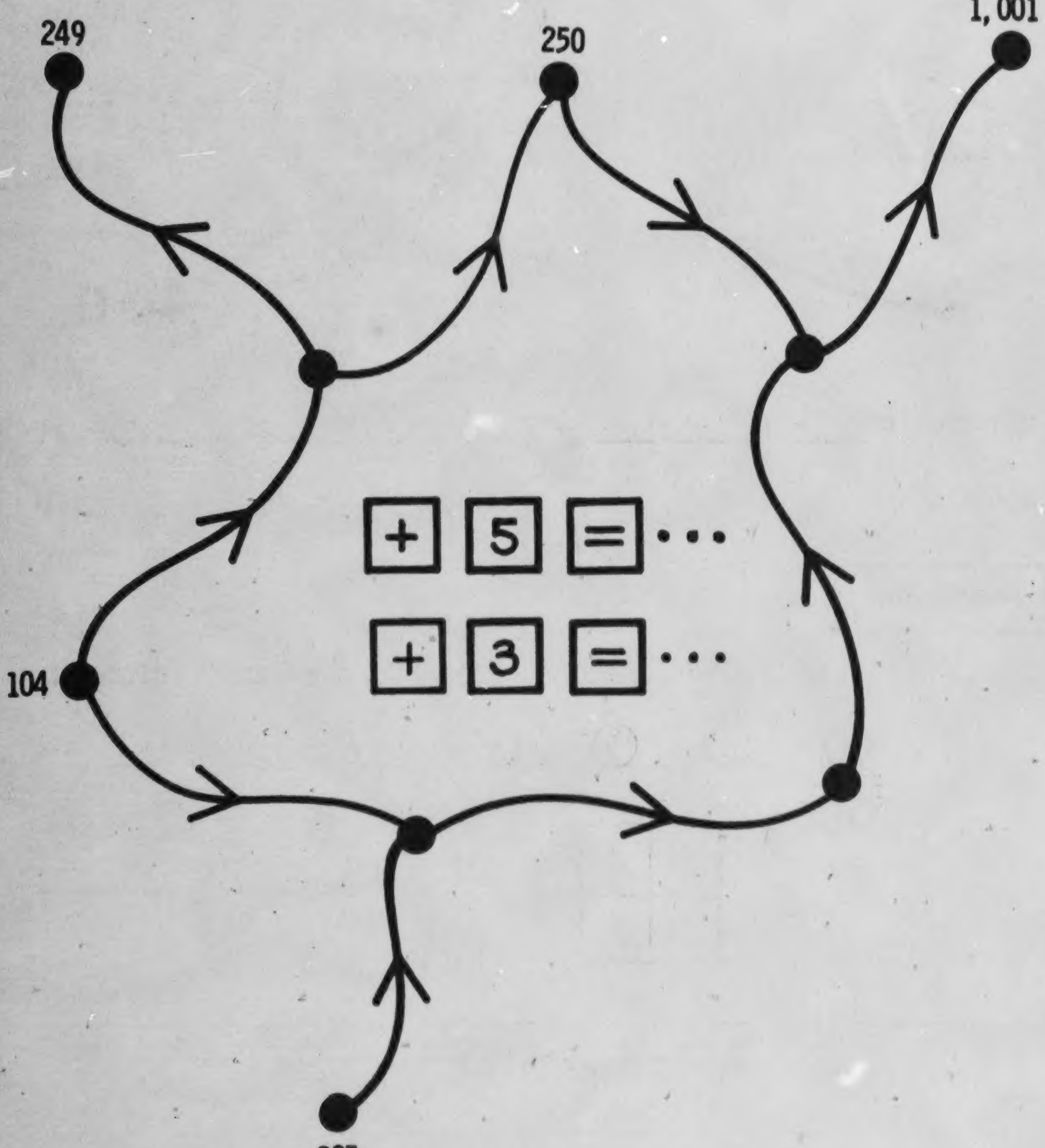
Third clue

Bim has exactly six positive divisors.

Bim is ________.

Name ______________________ **N 77 *****

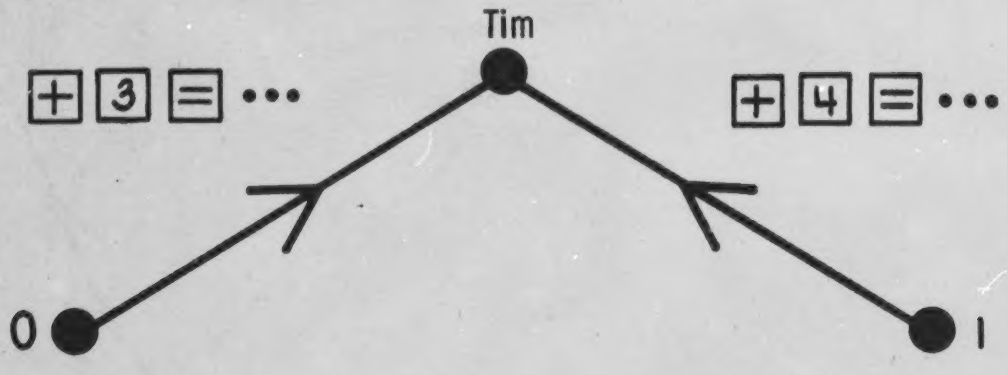
Label the dots. Many answers are possible.


249
250
1,001
104
205
+ 5 = ...
+ 3 = ...

Name ______________________

N 78 ****

First clue



Tim could be _____; _____; _____; _____; _____; _____; _____; _____;

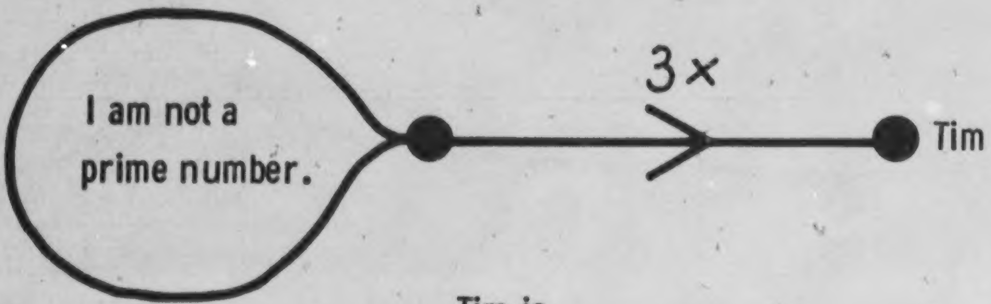
_____; and so on.

Second clue

Tim is between 200 and 240.

Tim is ______ or ______ or ______ or ______.

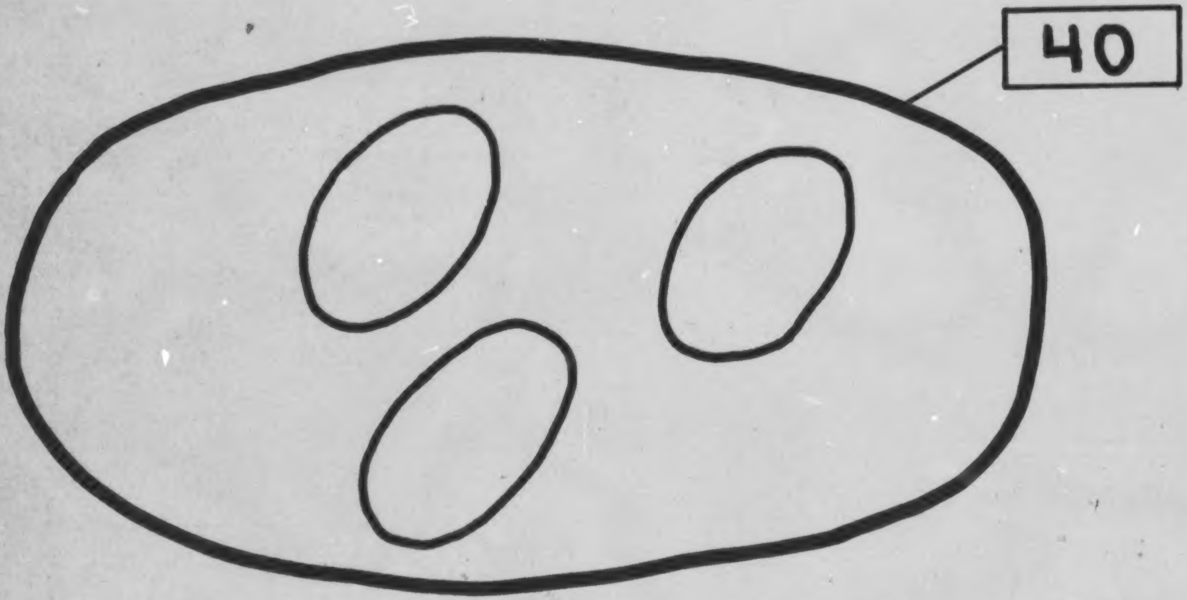
Third clue


I am not a prime number.
3×
Tim

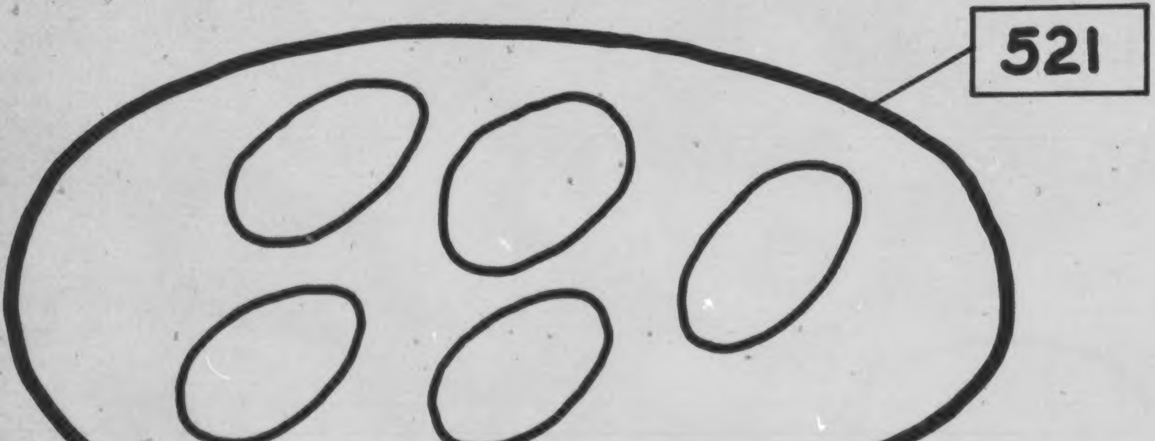
Tim is ______.

Name ______________________

N 79 *



Calculation: ________________ or ______________________

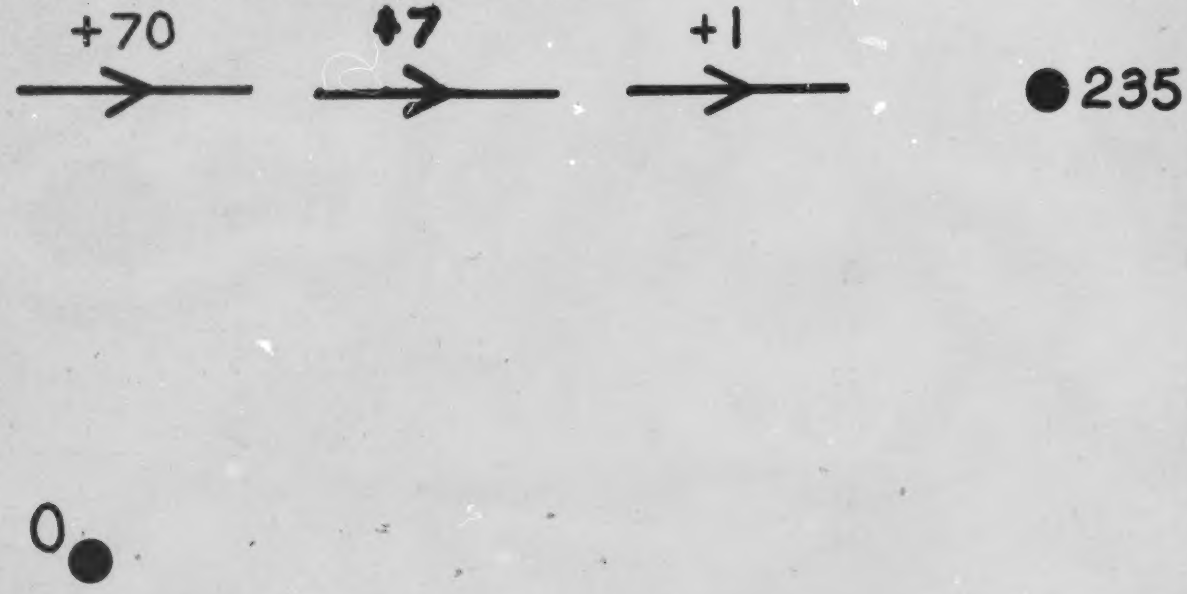


Calculation: ________________ or ______________________

Name ______________________

N 80 **

Build a blue-red-black road from 0 to 235.
Try to use as few arrows as possible.



235 tennis balls
Each can contains 7 tennis balls.

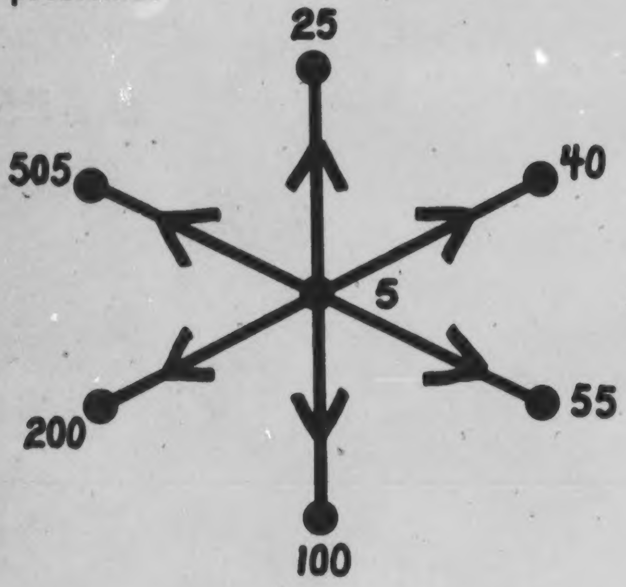
How many completely filled cans ? ____________

How many tennis balls left over ? ____________

N 81 *

Name ______________________________

Label each arrow "x ☐". Use the arrow pictures to solve the division problems.


25
505
40
5
200
55
100

25 ÷ 5 = ________

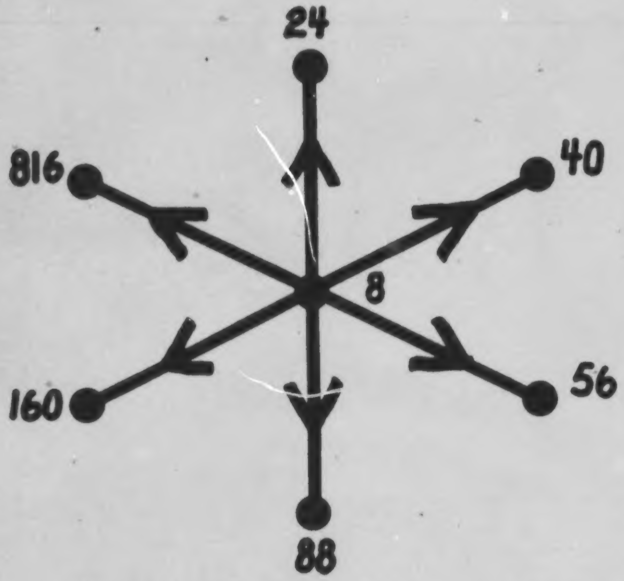
40 ÷ 5 = ________

55 ÷ 5 = ________

100 ÷ 5 = ________

200 ÷ 5 = ________

505 ÷ 5 = ________


24
816
40
8
160
56
88

24 ÷ 8 = ________

40 ÷ 8 = ________

56 ÷ 8 = ________

88 ÷ 8 = ________

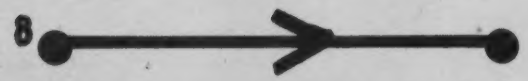
160 ÷ 8 = ________

816 ÷ 8 = ________

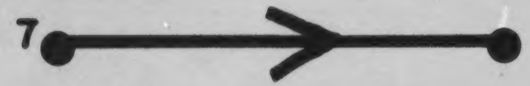
Name ____________________

N 82 **

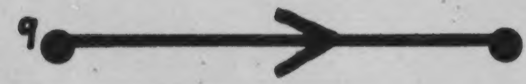
Label each arrow "x ☐". The ending number has to be the closest neighbor possible to the number in parentheses.
Use the arrow picture to solve the division problems.



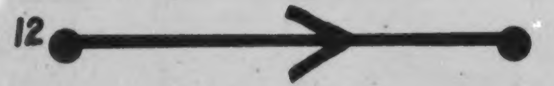
(90) 8)90 R = ____



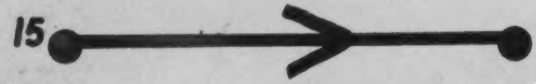
(150) 7)150 R = ____



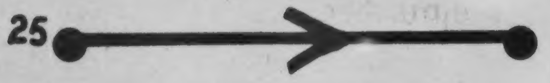
(200) 9)200 R = ____



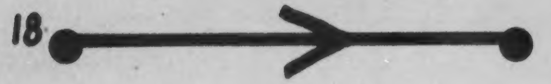
(280) 12)280 R = ____



(500) 15)500 R = ____



(1060) 25)1060 R = ____



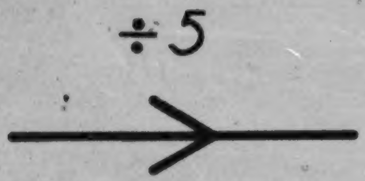
(2000) 18)2000 R = ____

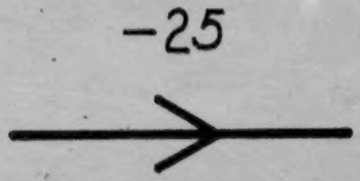
Name ______________________________

N 83 ***

First clue

Rank is the ending number of a red-blue road starting at 625 and using exactly two red arrows and two blue arrows.





625 ●

Rank is ______ or ______ or ______ or ______ or ______ or ______.

Second clue

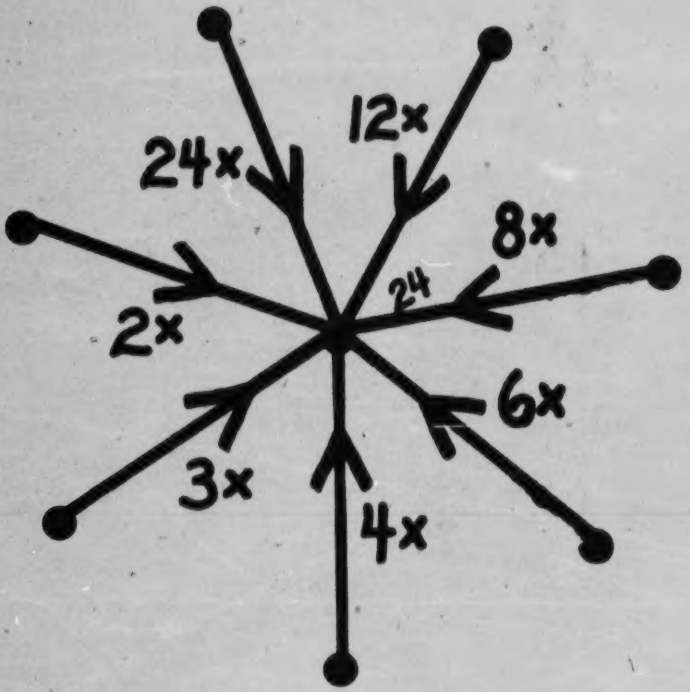
If you add 510 to Rank, you will find a square number.

Rank is ________.

N 84 *

Name ______________________________

Label the dots.


24x
12x
8x
24
2x
6x
3x
4x

Solve these problems.

$\frac{1}{2} \times 24 =$ ________

$\frac{1}{3} \times 24 =$ ________

$\frac{1}{4} \times 24 =$ ________

$\frac{1}{6} \times 24 =$ ________

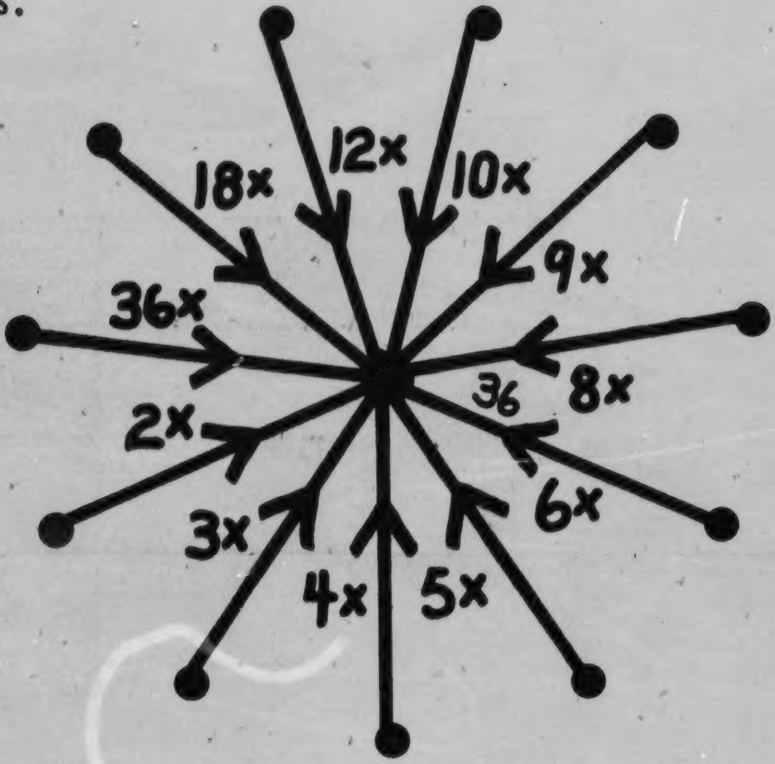
$\frac{1}{8} \times 24 =$ ________

$\frac{1}{12} \times 24 =$ ________

$\frac{1}{24} \times 24 =$ ________

Name ______________________ **N 85 ****

Label the dots.


12x
18x
10x
9x
36x
36
8x
2x
6x
3x
4x
5x

Solve these problems.

$\frac{1}{2}$ x 36 = ____ $\frac{1}{6}$ x 36 = ____ $\frac{1}{12}$ x 36 = ____

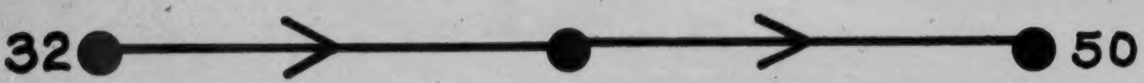
$\frac{1}{3}$ x 36 = ____ $\frac{1}{8}$ x 36 = ____ $\frac{1}{18}$ x 36 = ____

$\frac{1}{4}$ x 36 = ____ $\frac{1}{9}$ x 36 = ____ $\frac{1}{36}$ x 36 = ____

$\frac{1}{5}$ x 36 = ____ $\frac{1}{10}$ x 36 = ____

Name ________________________________

N 86 ***



Complete the table.

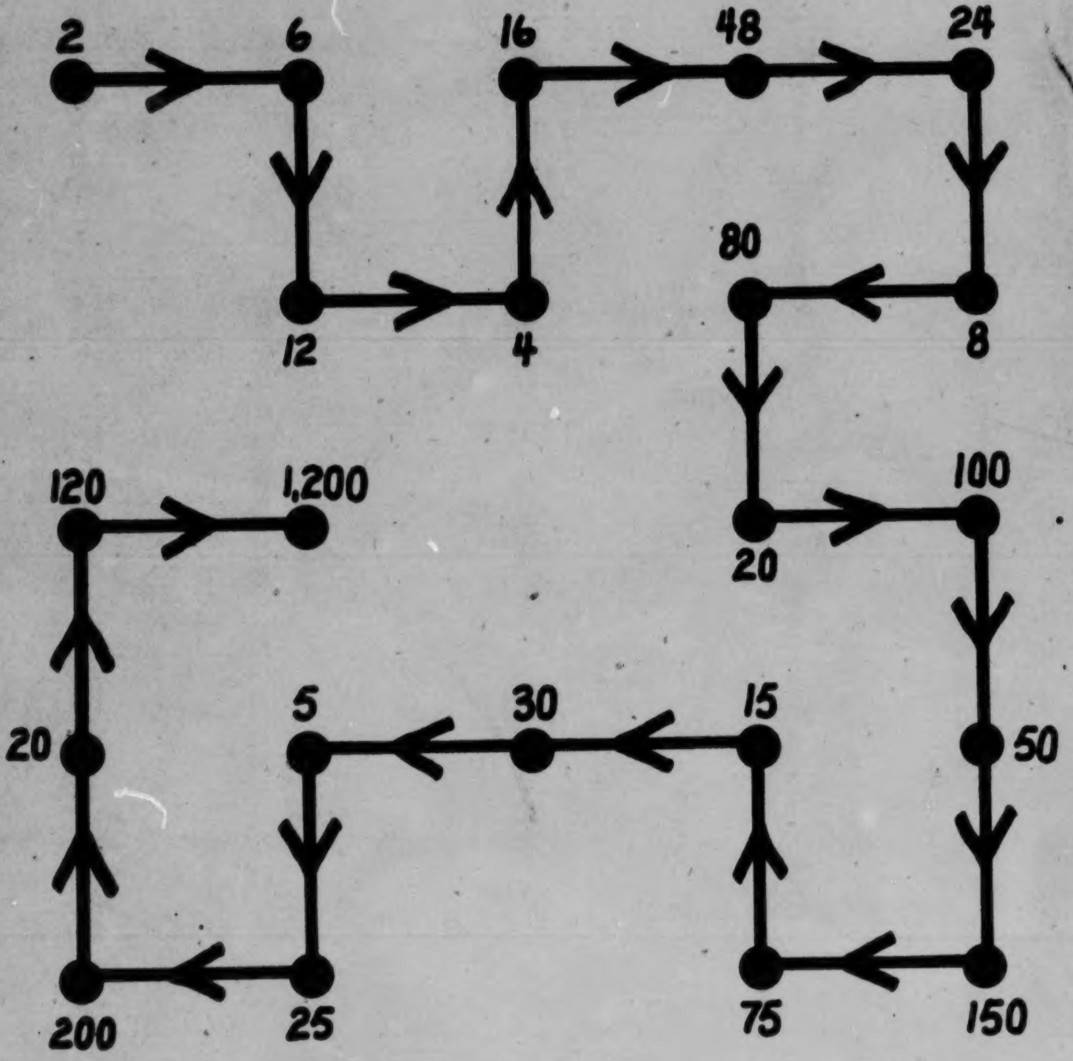
$\longrightarrow$	$\longrightarrow$
$\frac{1}{2}$ x	
$\frac{1}{4}$ x	
$\frac{1}{8}$ x	
$\frac{1}{10}$ x	
$\frac{1}{16}$ x	
$\frac{1}{32}$ x	
$\frac{1}{5}$ x	
$\frac{1}{64}$ x	
$\frac{1}{100}$ x	
−40	

$\longrightarrow$	$\longrightarrow$
	$\frac{1}{2}$ x
	$\frac{1}{3}$ x
	$\frac{1}{4}$ x
	$\frac{1}{5}$ x
	$\frac{1}{6}$ x
	$\frac{1}{10}$ x
	$\frac{1}{20}$ x
	$\frac{1}{100}$ x
	+100
	−100

N 87 *

Name ____________________

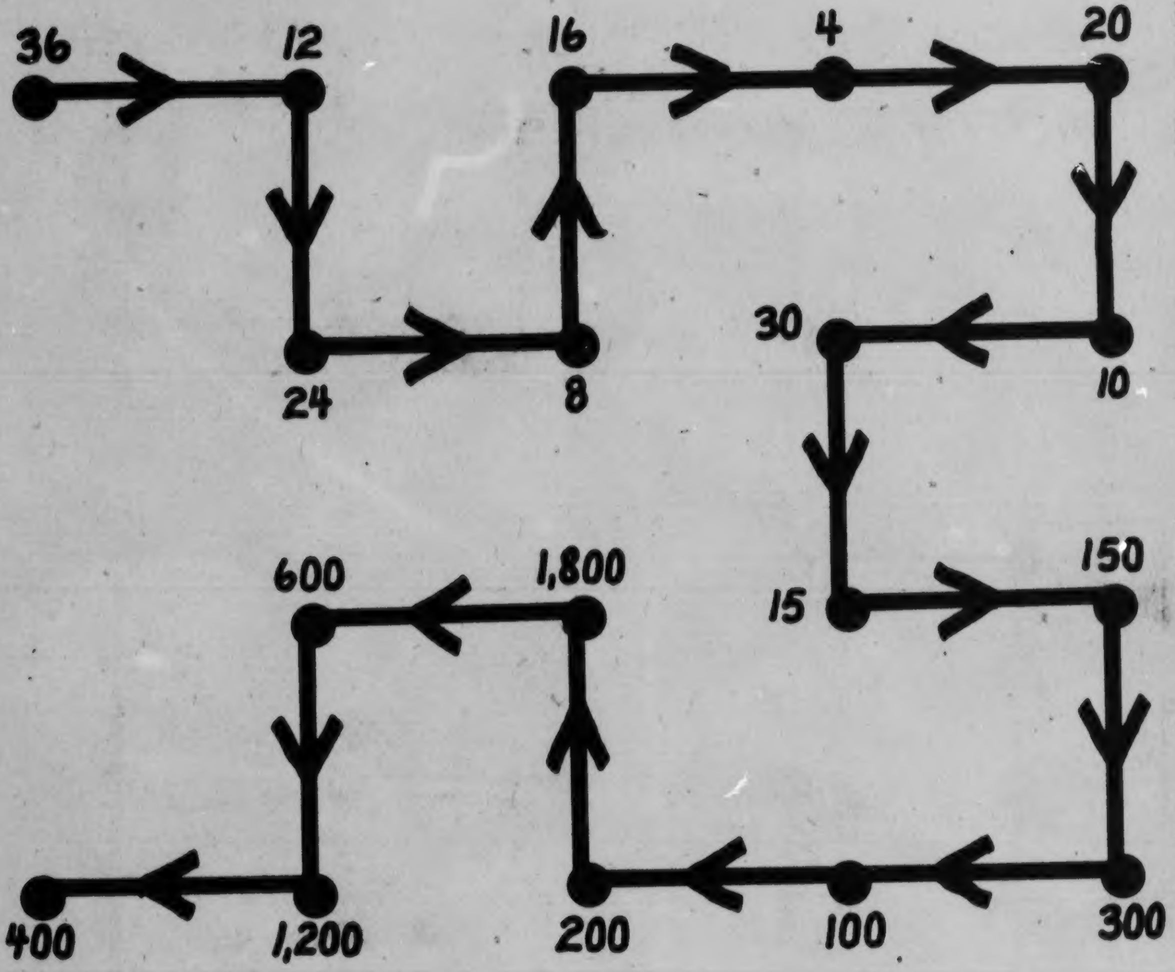
Label each arrow "x ☐" or " ÷☐".


2
6
12
4
16
48
24
8
80
20
100
50
150
75
15
30
5
25
200
20
120
1,200

N 88 **

Name ____________________

Label each arrow "x ☐" or "÷ ☐".


36
12
16
4
20
24
8
30
10
15
150
600
1,800
400
1,200
200
100
300

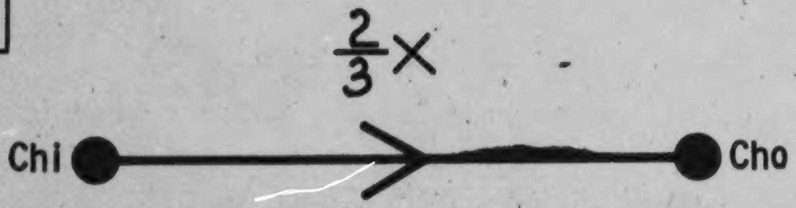
Use blue to draw the nine $\frac{2}{3}$x arrows in this picture.

Name ______________________________

N 89 ***

Cho is a secret WHOLE number.

First clue



Chi and Cho could be: (Some are done for you.)

Chi	Cho	Chi	Cho	Chi	Cho	Chi	Cho
___	___	___	___	10.5	7	___	___
___	___	___	___	___	___	___	___
___	___	24	16	___	___	___	___

Second clue

Chi is a multiple of 9 between 50 and 100.

Chi and Cho could be:

Chi	Cho	Chi	Cho	Chi	Cho
___	___	___	___	___	___
___	___	___	___	___	___

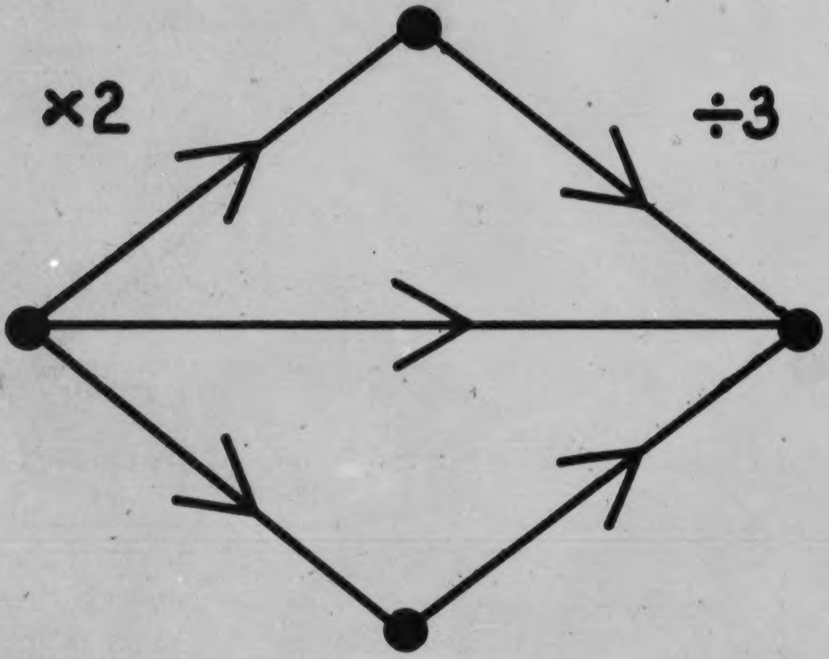
Third clue

Cho is a square number.

Chi is ________. Cho is ________.

Name ______________________________

N 90 *



Label the blue arrow and solve these problems.

$\frac{2}{3} \times 6 =$ __________

$\frac{2}{3} \times 60 =$ __________

$\frac{2}{3} \times 9 =$ __________

$\frac{2}{3} \times 12 =$ __________

$\frac{2}{3} \times 15 =$ __________

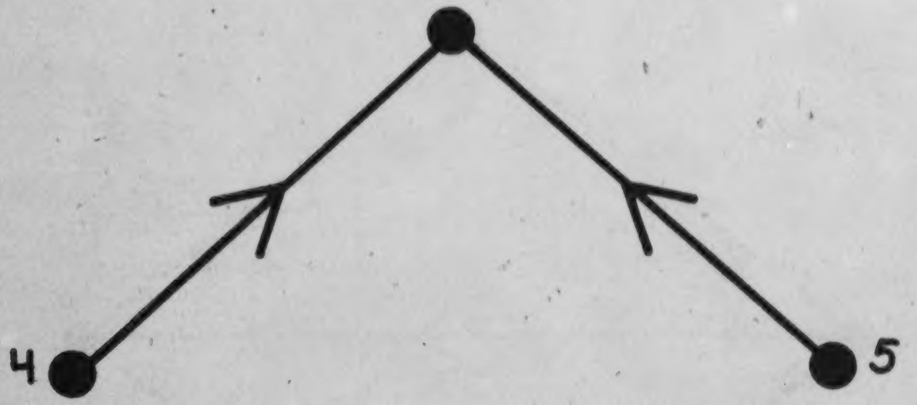
$\frac{2}{3} \times 24 =$ __________

$\frac{2}{3} \times 30 =$ __________

$\frac{2}{3} \times 45 =$ __________

N 91 **

Name ______________________________



Complete these charts by labeling the red arrow and the blue arrow "x ☐".

→	→

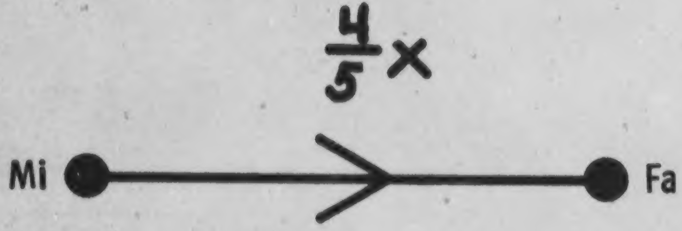
→	→

Name ______________________________

N 92 ***

Mi and Fa are two secret WHOLE numbers.

First clue



Mi and Fa could be:

Mi	Fa	Mi	Fa	Mi	Fa
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	and so on.	

Mi + Fa could be ______; ______; ______; ______; ______; ______;

______; ______; and so on.

What do you notice about Mi + Fa ? Write your answer below.

____

Second clue

Mi + Fa = 81.

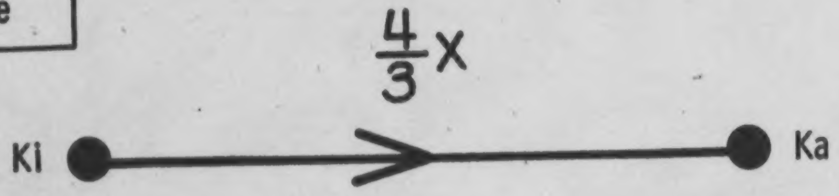
Mi is ______________. Fa is ______________.

Name ____________________________

N 93 ****

Ki and Ka are two secret WHOLE numbers.

First clue



Ki and Ka could be:

Ki	Ka
27	
	40

Ki	Ka
36	
	100
72	

Ki	Ka
	88
99	

and so on.

Ki + Ka could be _____; _____; _____; _____; _____; _____; _____; _____;

_____; _____; _____; _____; and so on.

What do you notice about Ki + Ka ? Write your answer below.

____

Second clue

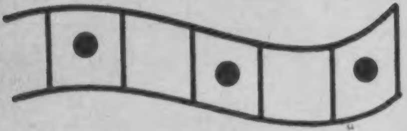
Ki + Ka is between 100 and 110.

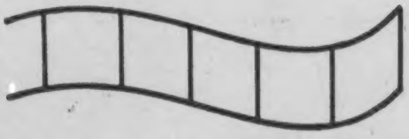
Ki + Ka = _________. Ki is _________. Ka is _________.

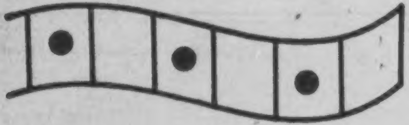
Name ______________________

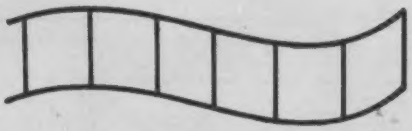
N 94 *

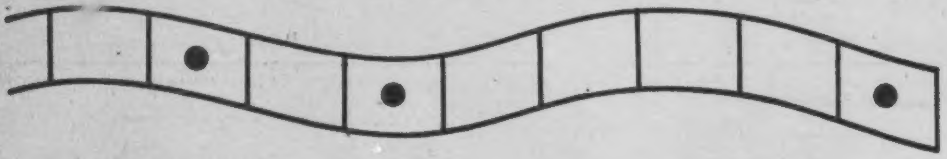
ON THE BINARY SNAKE

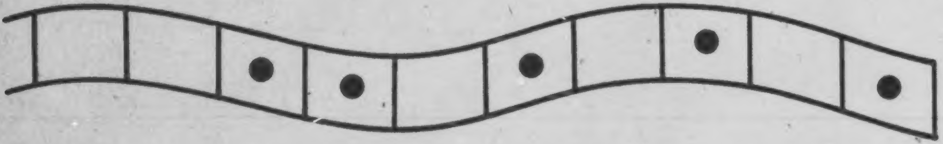

= ___

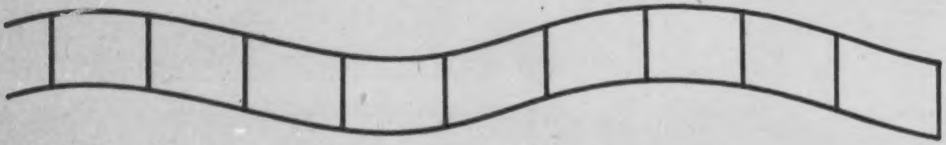

= 20


= ___


= 40


= ___


= ___


= 100

N 95 **

Name ______________________

Last Sunday, Nabu wrote "30" in his notebook.

Which friends did he invite? ______________________

		h	g	f	e	d	c	b	a

On another day he wrote "75" in his notebook.

Which friends did he invite? ______________________

		h	g	f	e	d	c	b	a

The day Nabu wrote "150" in his notebook, which friends did he invite?

____

		h	g	f	e	d	c	b	a

The day Nabu wrote "200" in his notebook, which friends did he invite?

____

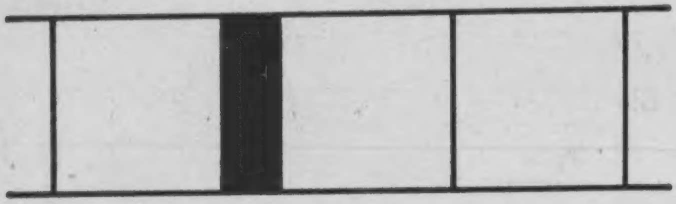
		h	g	f	e	d	c	b	a

Name ______________________

N 96 ***

First clue

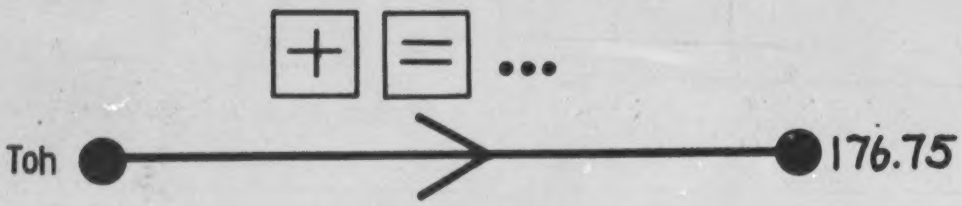
Toh can be put on this part of the binary abacus using at most one checker on each board.



Toh is ______ or ______ or ______ or ______ or ______ or ______

or ______ or ______.

Second clue



Toh is ________ or ________.

Third clue

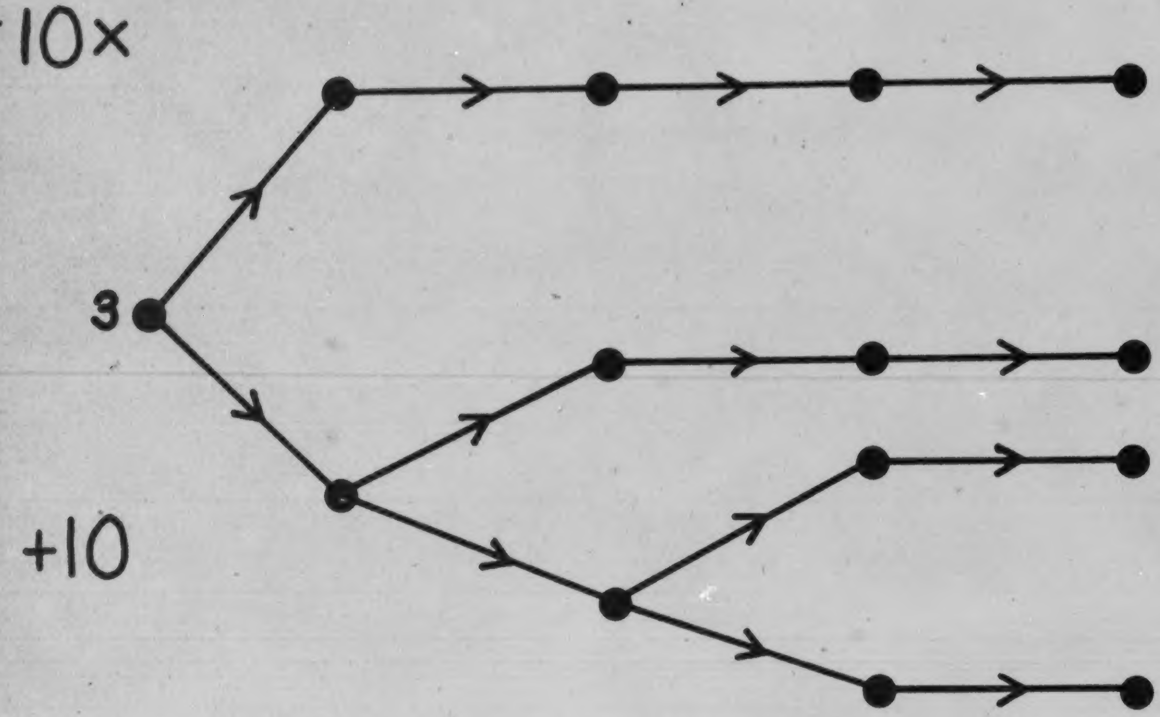
If you multiply Toh by 28, you will find a square number.

Toh is ________.

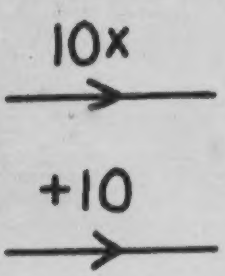
Name ______________________

N 97 *

Label the dots.



How many roads are there using one red arrow and three blue arrows? ______



Deck is the ending number of a red-blue road starting at 3 and using one red arrow and three blue arrows.

Deck could be ________, ________, ________, or ________.

Name ______________________ **N 98 ****

Draw a picture showing all the red-blue roads using one red arrow and two blue arrows.

How many roads did you find? __________

$\xrightarrow{4x}$

$\xrightarrow{+5}$

Chex is the ending number of a red-blue road starting at 3 and using one red arrow and two blue arrows.

What numbers could Chex be? __________

Name ________________________________

N 99 ***

Draw a picture showing all the red-blue roads using two red arrows and three blue arrows.

How many roads using two red arrows and three blue arrows? ________

$\xrightarrow{5\times}$

$\xrightarrow{+1}$

Tex is the ending number of a red-blue road starting at 10 and using two red arrows and three blue arrows.

Tex could be: ______________________________

Name ______________________________

N 100 ****

How many roads using three red arrows and two blue arrows are there? _____

If you want to check your answer, draw a picture showing all the roads.

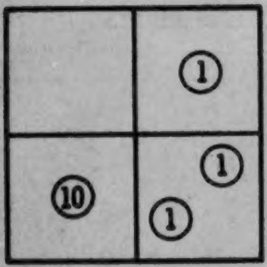
How many roads using three red arrows and three blue arrows are there? _____

If you want to check your answer, draw a picture showing all the roads. (Use a separate sheet of paper.)

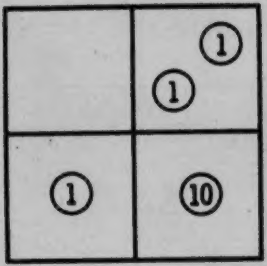
N 101 *

Name ______________________

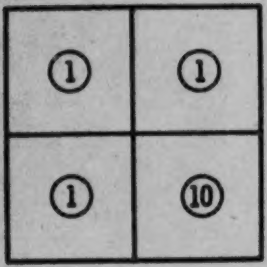
Complete:



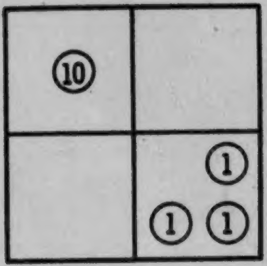
= ______



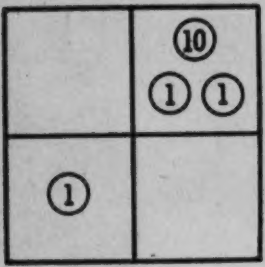
= ______



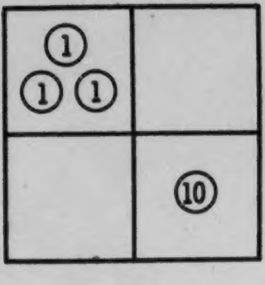
= ______



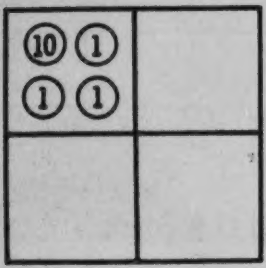
= ______



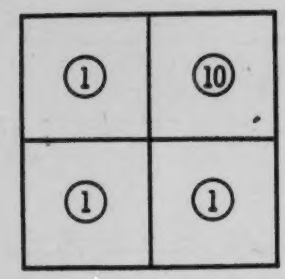
= ______



= ______



= ______



= ______

Name ______________________

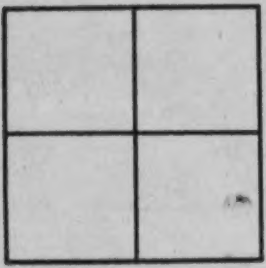
N 102 *

Write any number you wish on the Minicomputer using exactly one dime and three pennies.

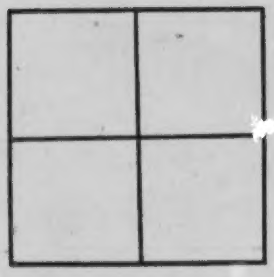
Name ______________________

N 103 **

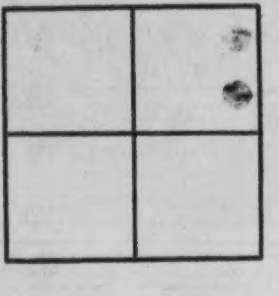
Write these numbers on the Minicomputer using exactly one dime and three pennies.



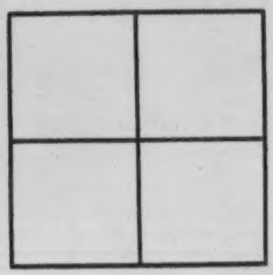
= 22



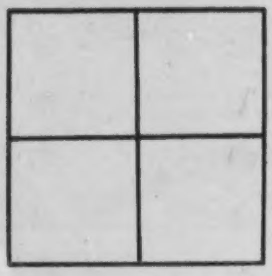
= 47



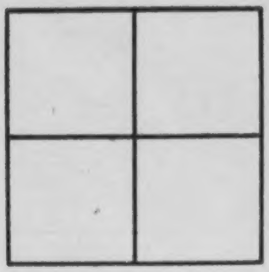
= 30



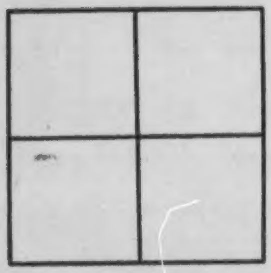
= 100



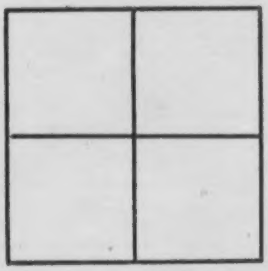
= 51



= 94



= 40

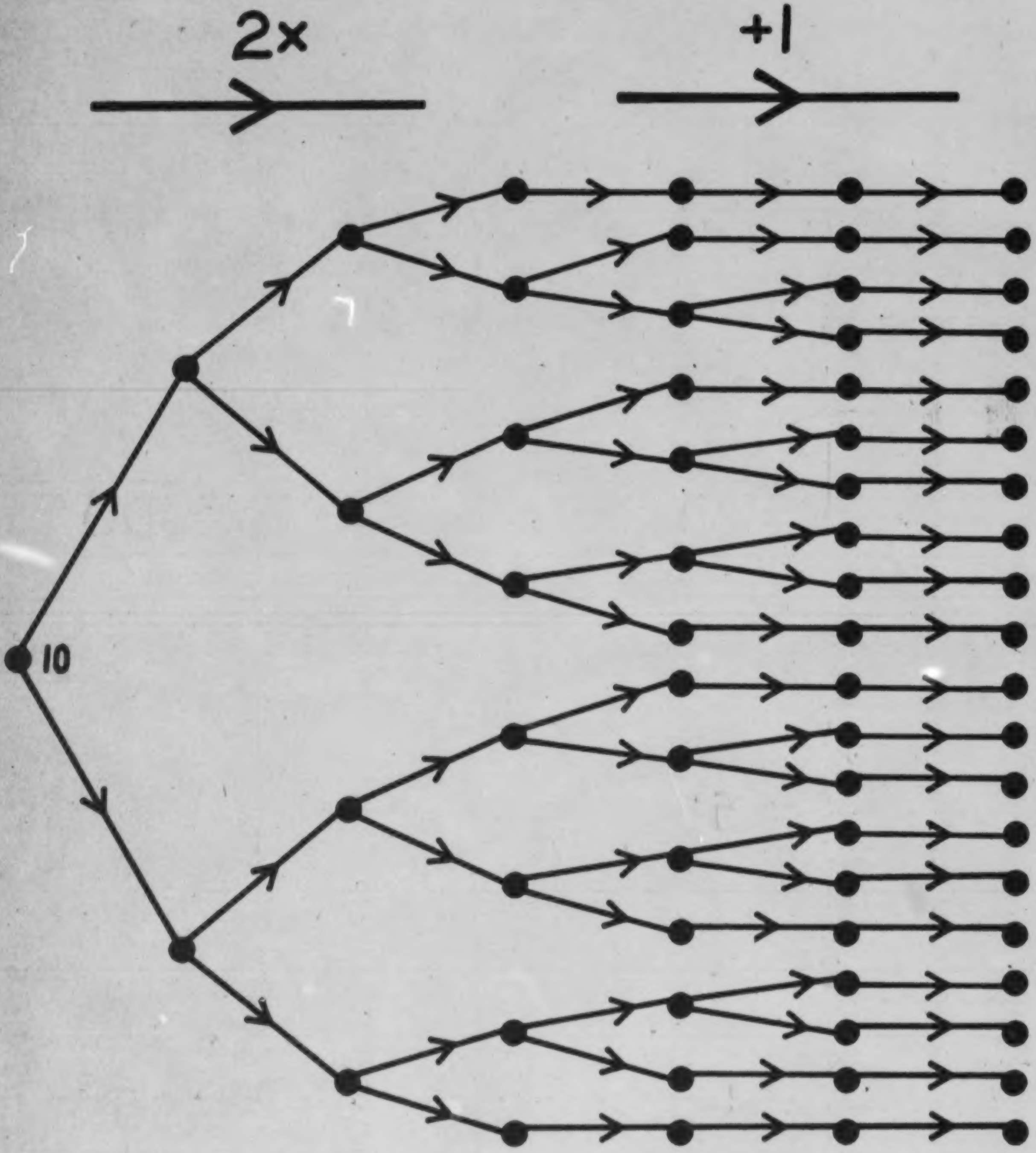


= 45

Name ____________________

N 104 *

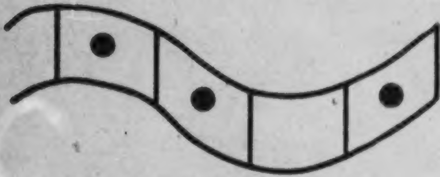
Label all the dots.



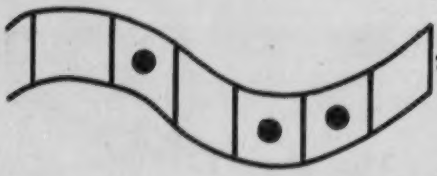
Name ______________________________

N 105 *

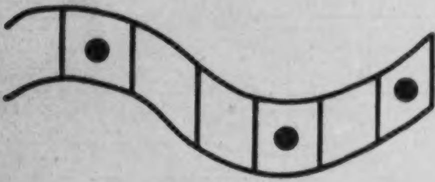
ON THE BINARY SNAKE



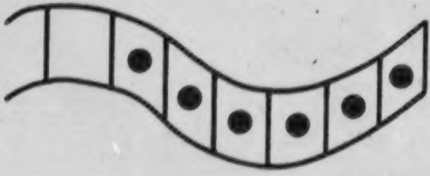
= ____



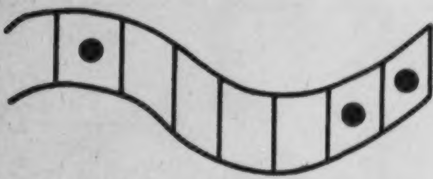
= ____



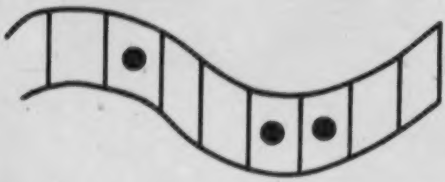
= ____



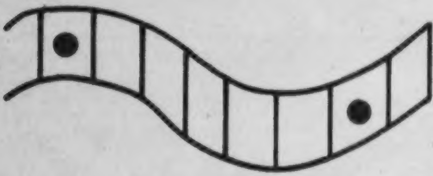
= ____



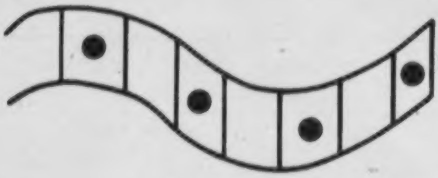
= ____



= ____



= ____

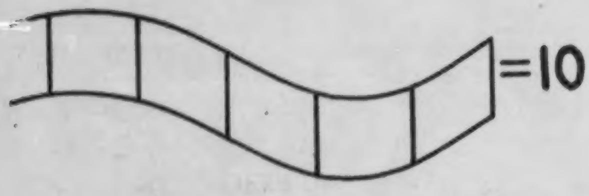


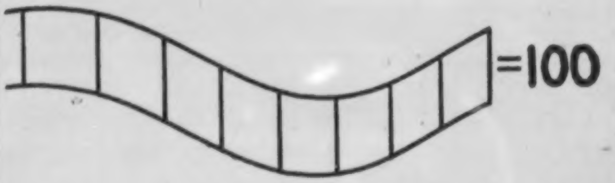
= ____

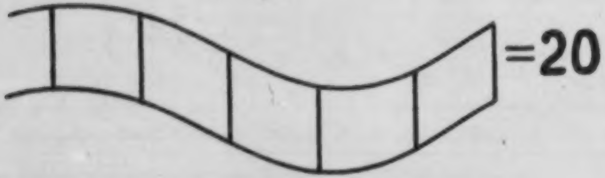
N 106 **

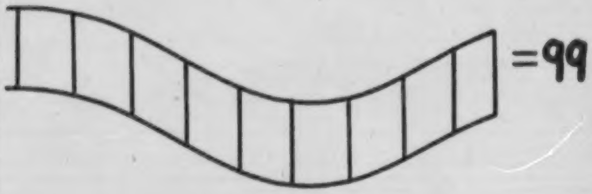
Name ______________________

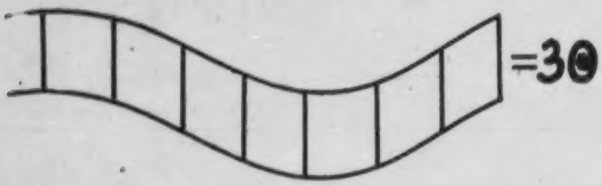
Put each number on the binary snake. Use at most one checker on each board.

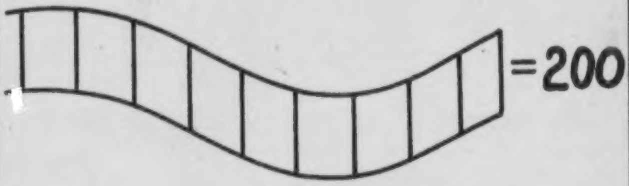


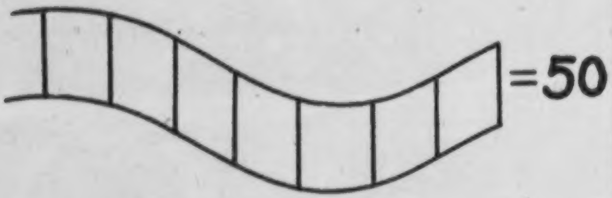


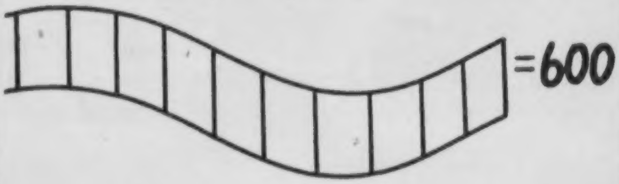












Name ______________________________

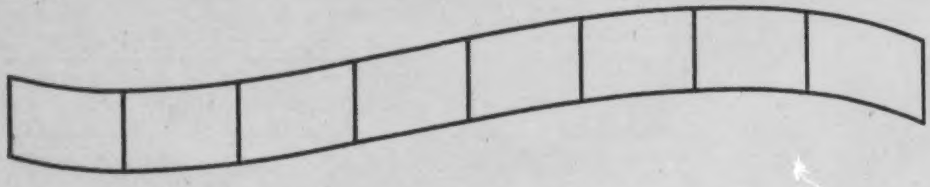
N 107 ***

First clue Clip is a positive prime number.

Clip could be _____; _____; _____; _____; _____; _____; _____; _____;

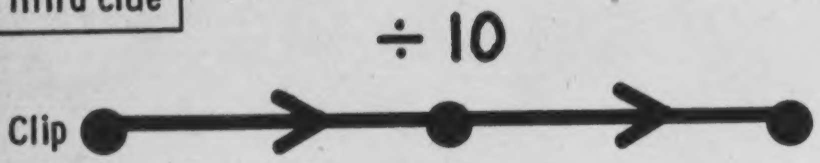
_____; _____; _____; _____; _____; _____; _____; _____; and so on.

Second clue Clip can be put on this binary snake using exactly two checkers (positive or negative).



Clip is _____ or _____ or _____ or _____ or _____ or _____ or _____.

Third clue



I am between 0.15 and 1.5

Clip is _____ or _____ or _____.

Fourth clue If you multiply Clip by himself, you will find a number larger than 1,000.

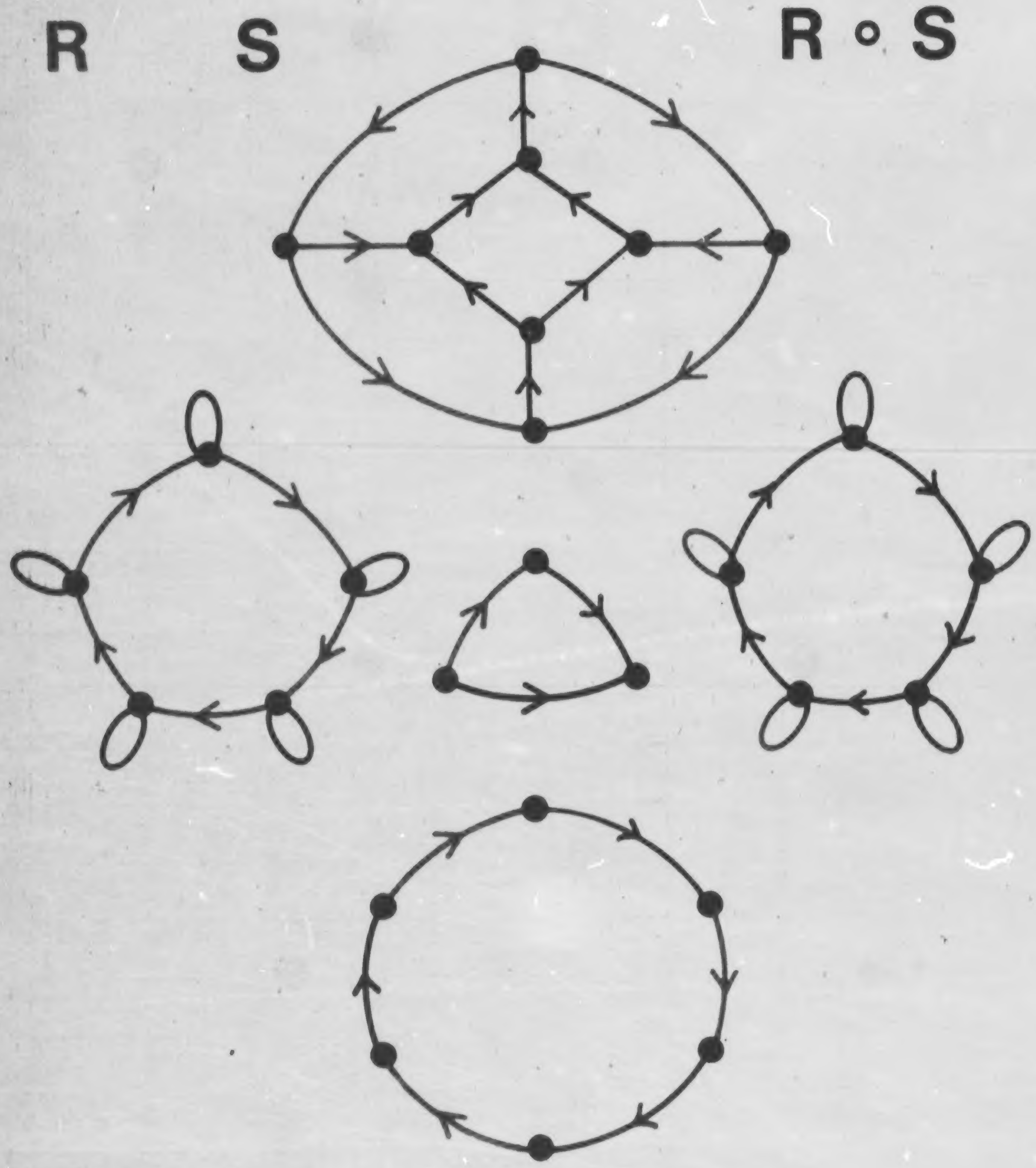
Clip is _____.

101

L 01 *

Name ______________________________

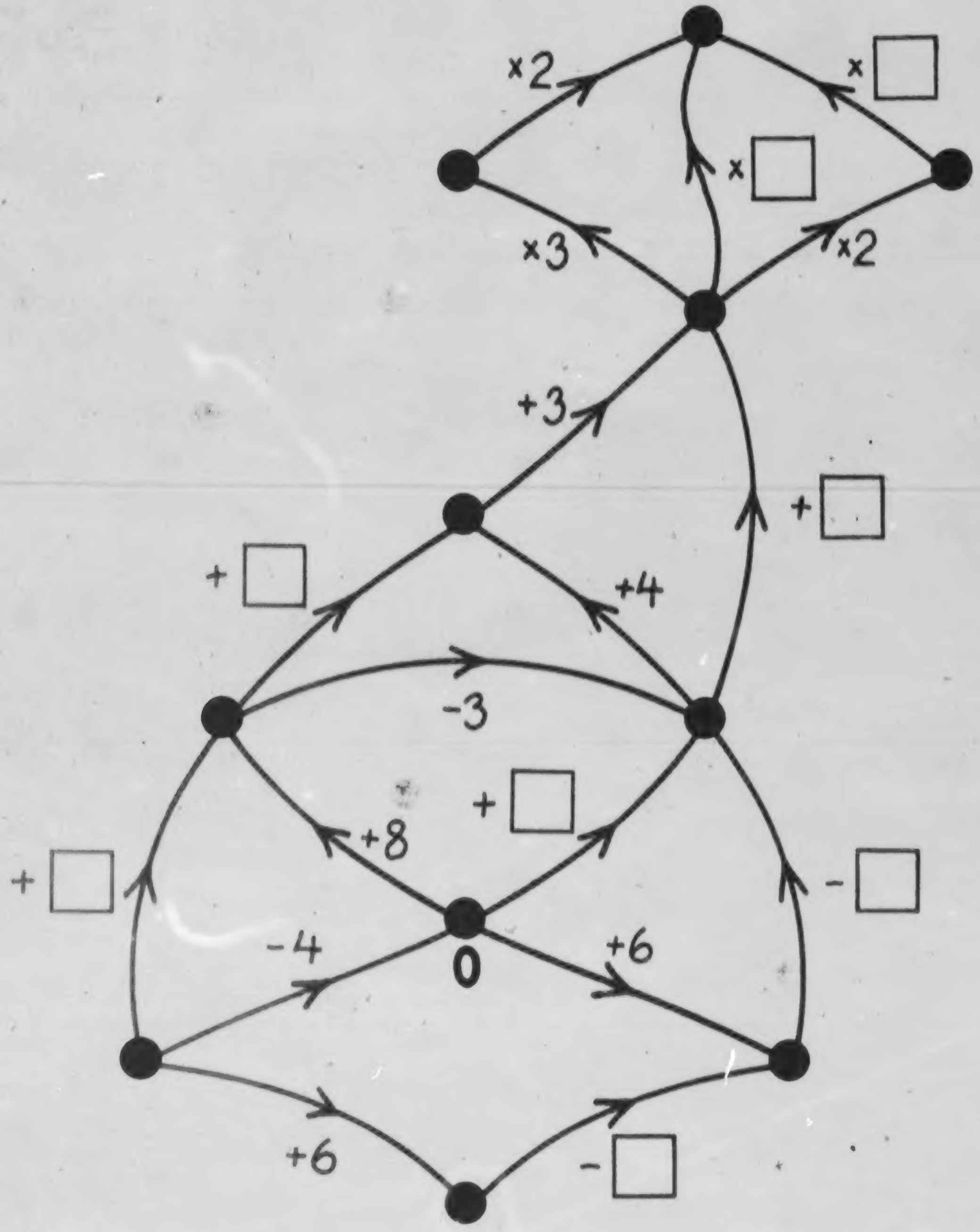
Draw the missing red arrows.



L 02 **

Name ____________________

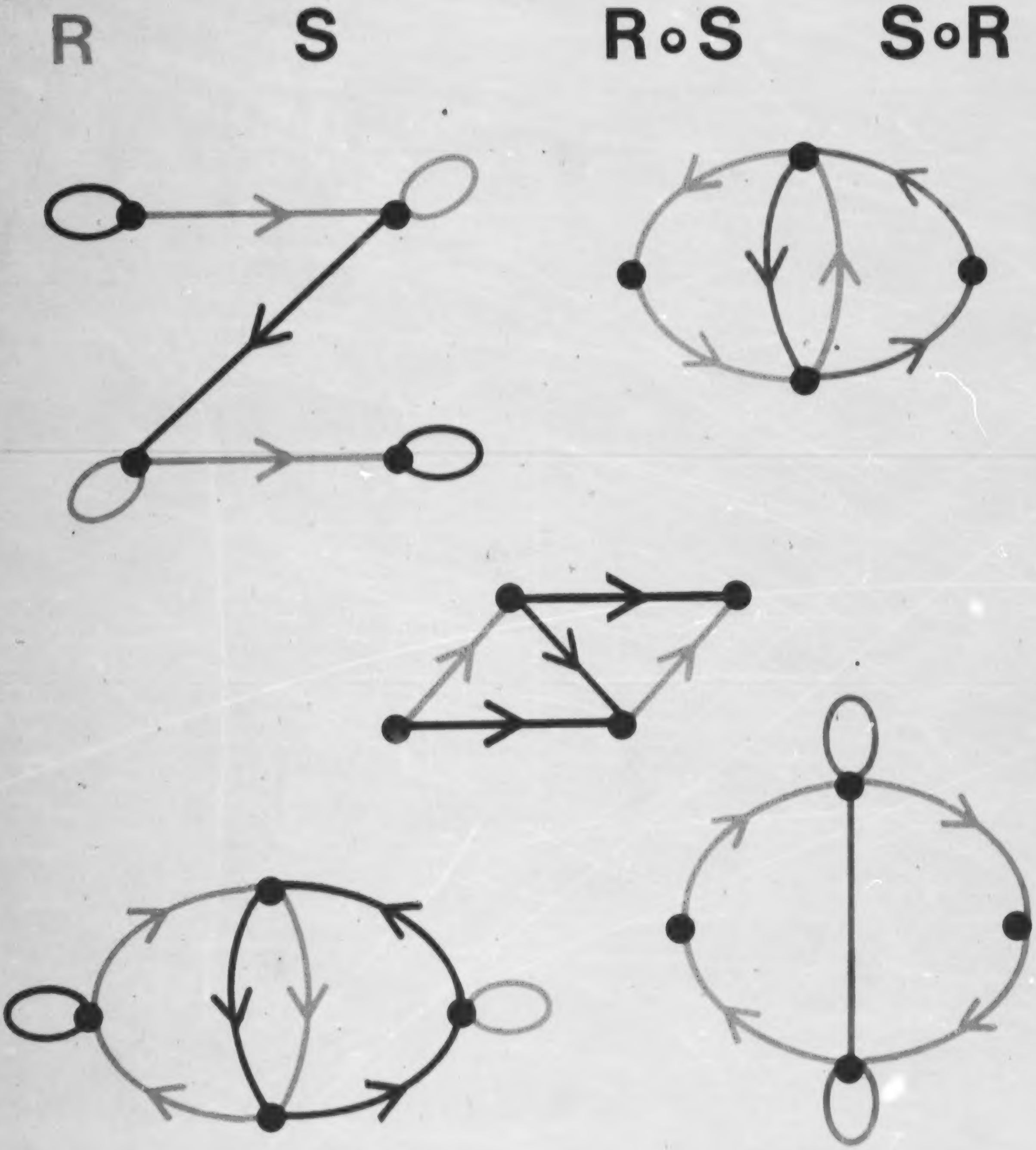
Label the dots and arrows.


×2
× ☐
× ☐
×3
×2
+3
+ ☐
+ ☐
+4
-3
+ ☐
+8
+ ☐
- ☐
-4
0
+6
+6
- ☐

L 03 ***

Name ______________________________

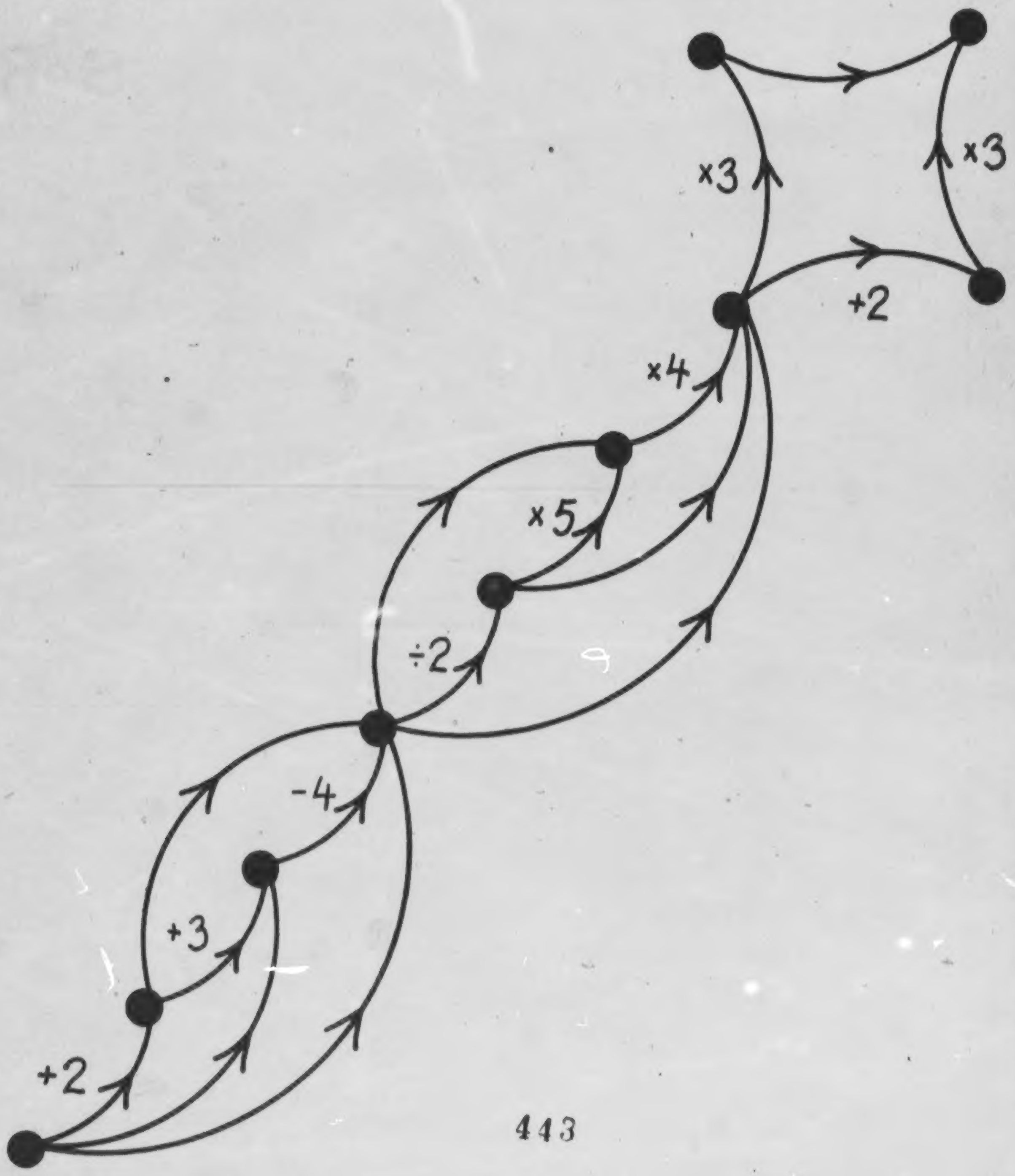
Draw the missing red and black arrows, loops, and cords.



L 04 ****

Name ______________________________

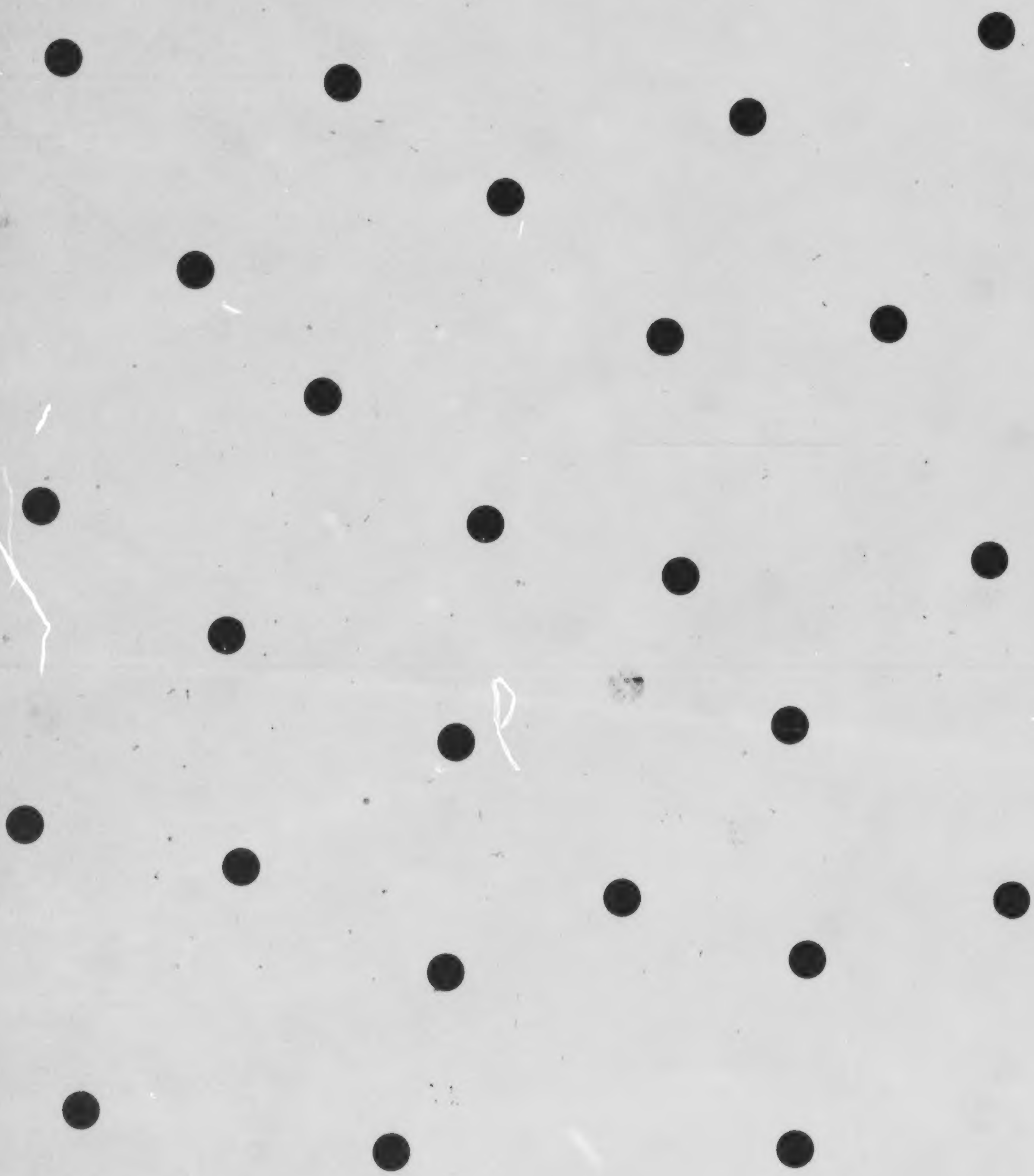
Label the arrows.



Name ______________________

L 05 *

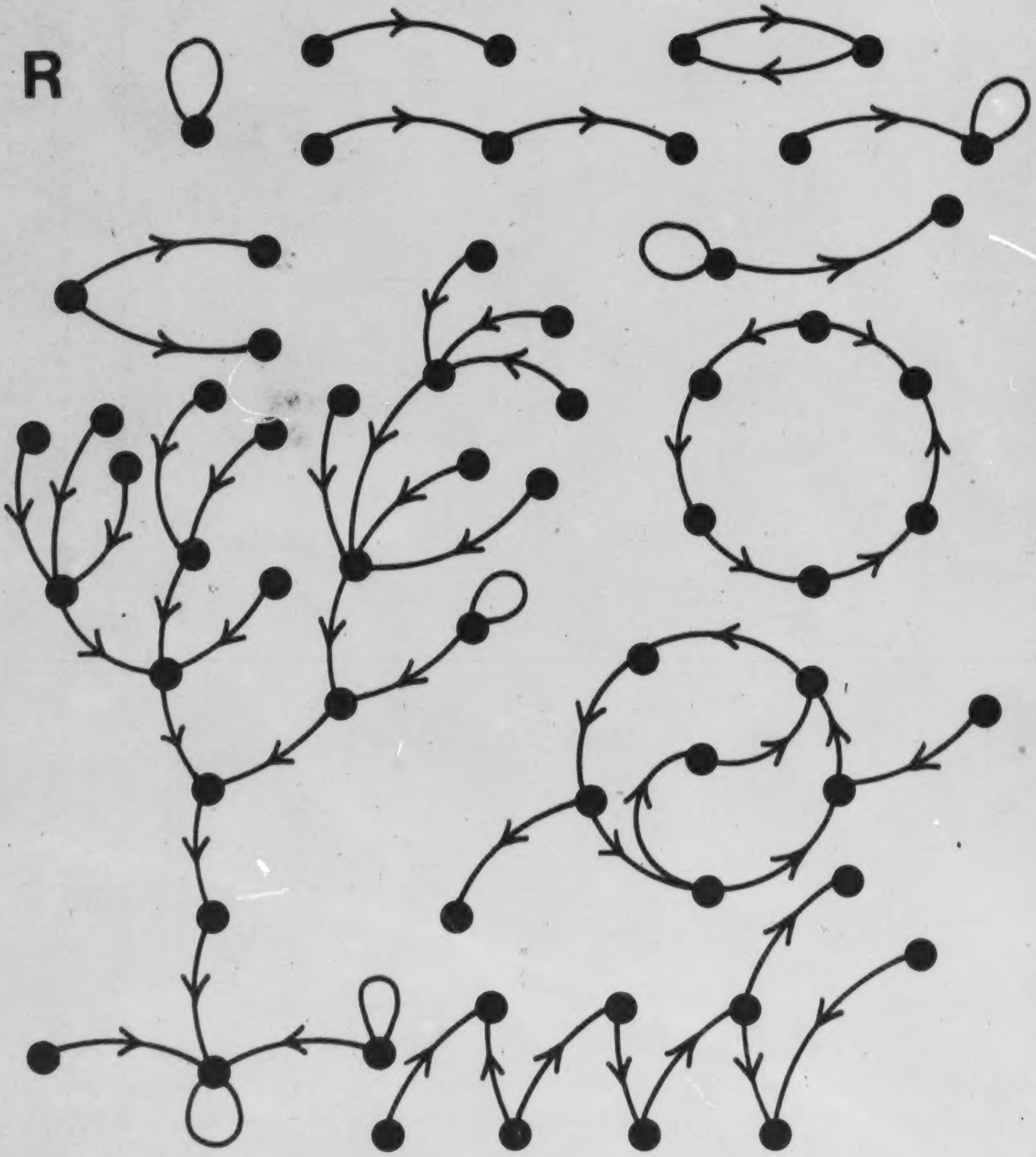
Draw a relation in blue such that at most one blue arrow starts at each dot.



Name ________________________

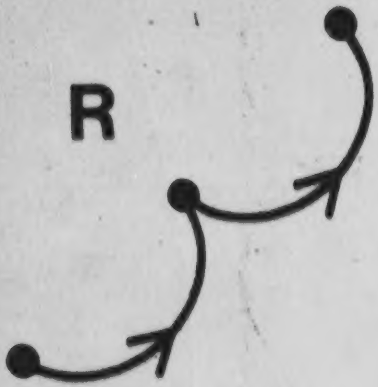
L 06 *

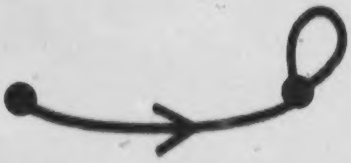
R is not a function. Circle all the guilty dots in red.

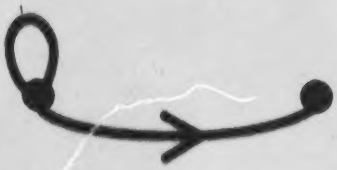


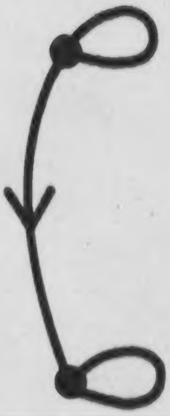
Name ______________________ **L 07 ***

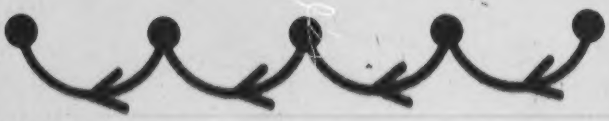
Draw all the missing red arrows and loops.

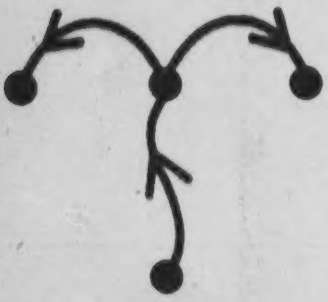


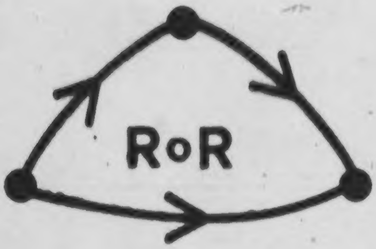


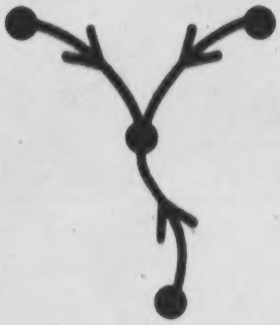


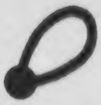


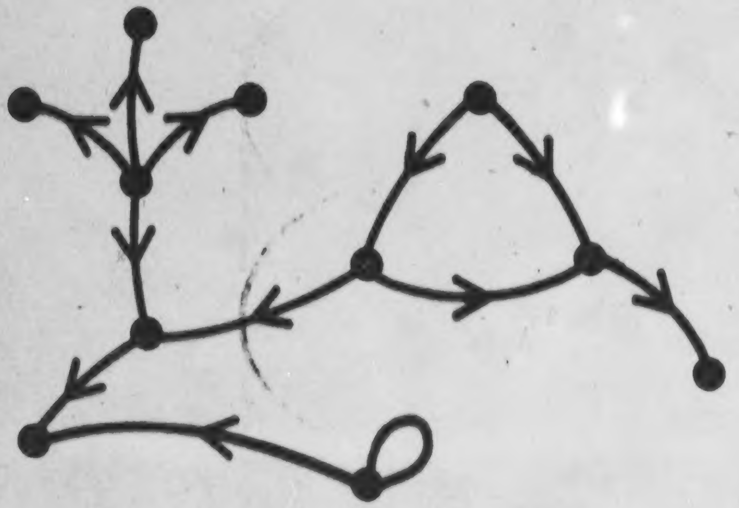


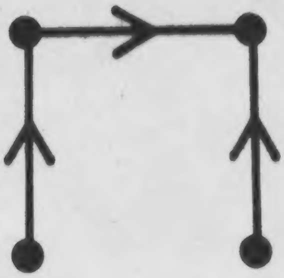








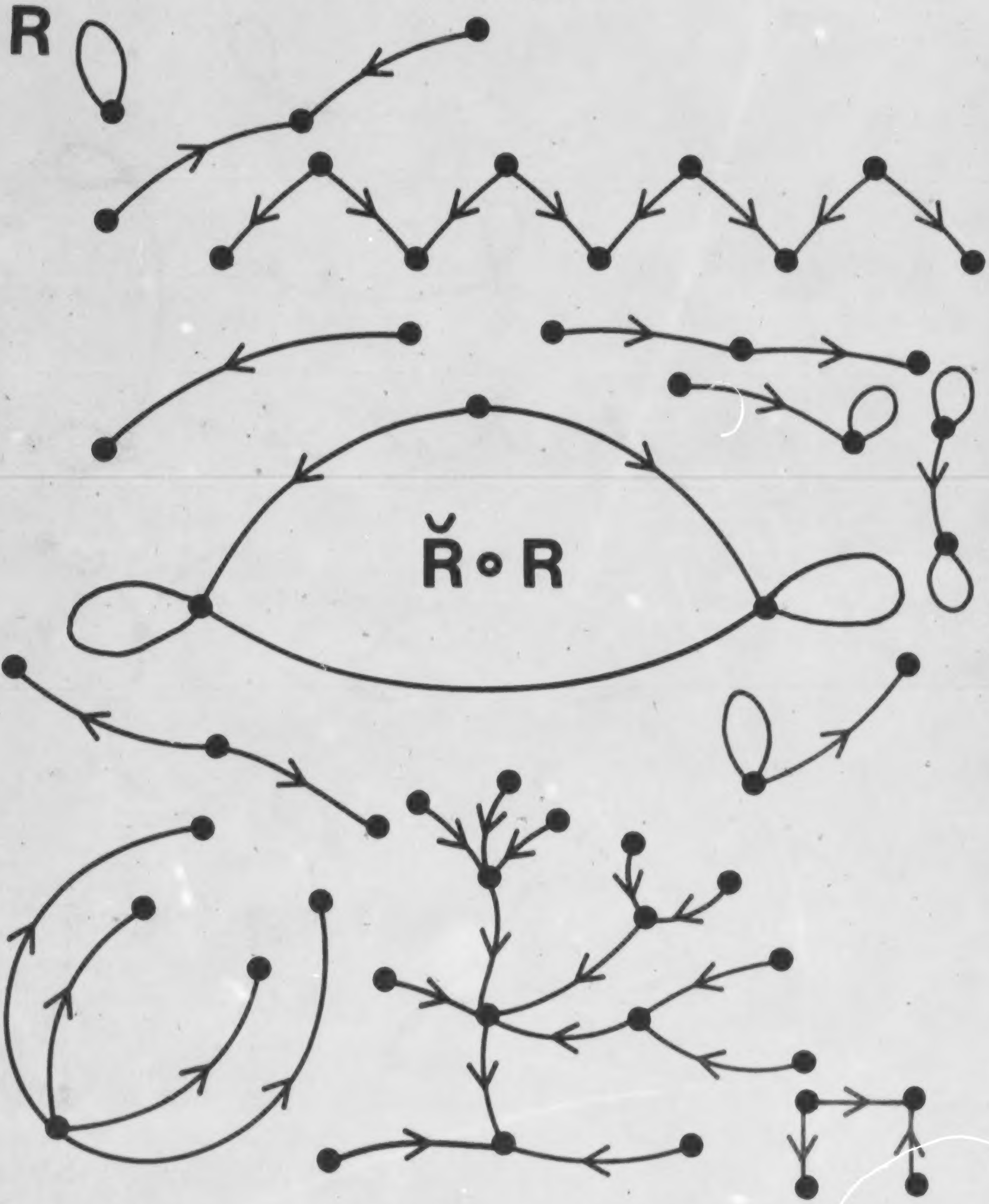




Name ______________________________

L 08 **

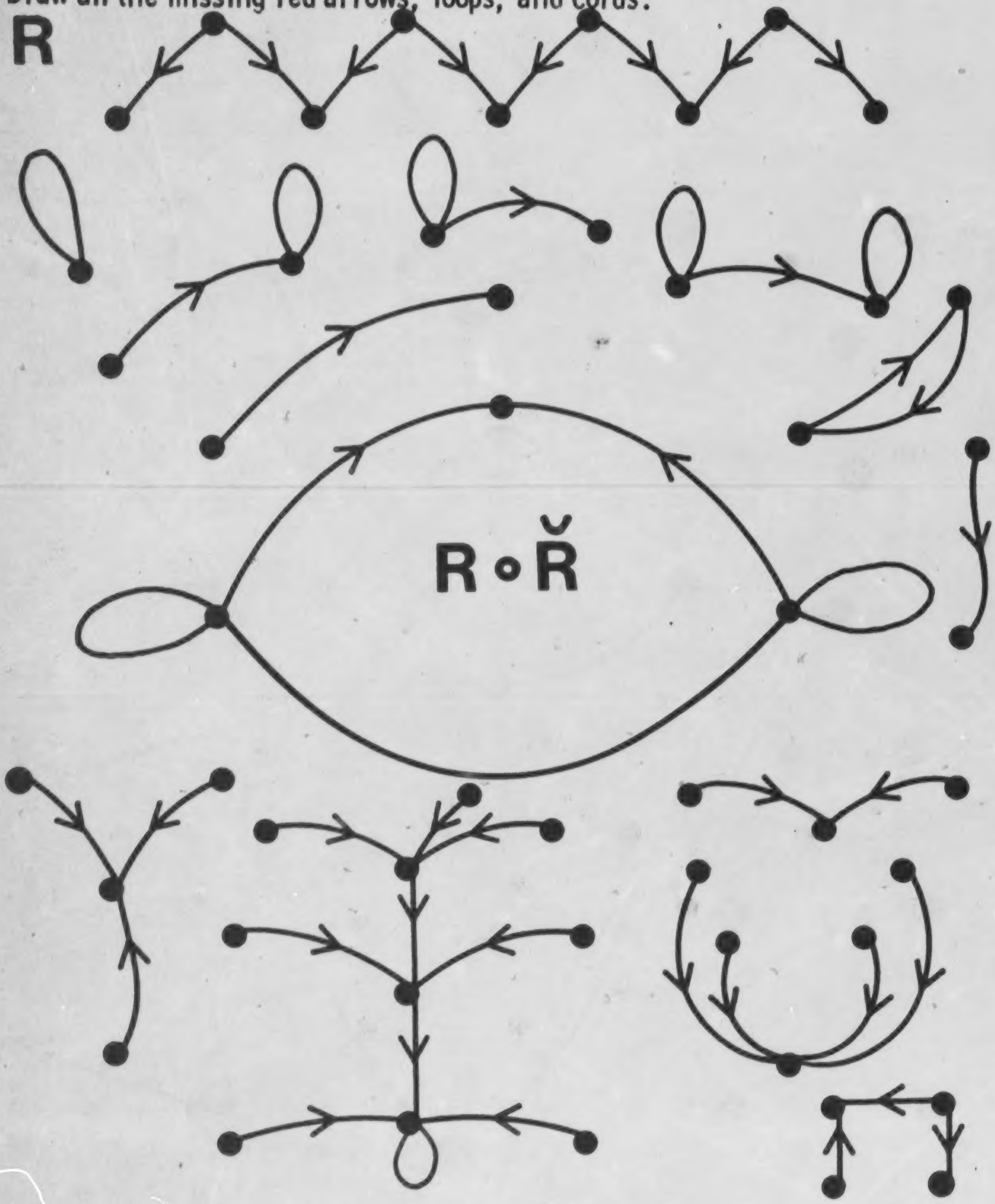
Draw all the missing red arrows, loops, and cords.



L 09 **

Name ______________________________

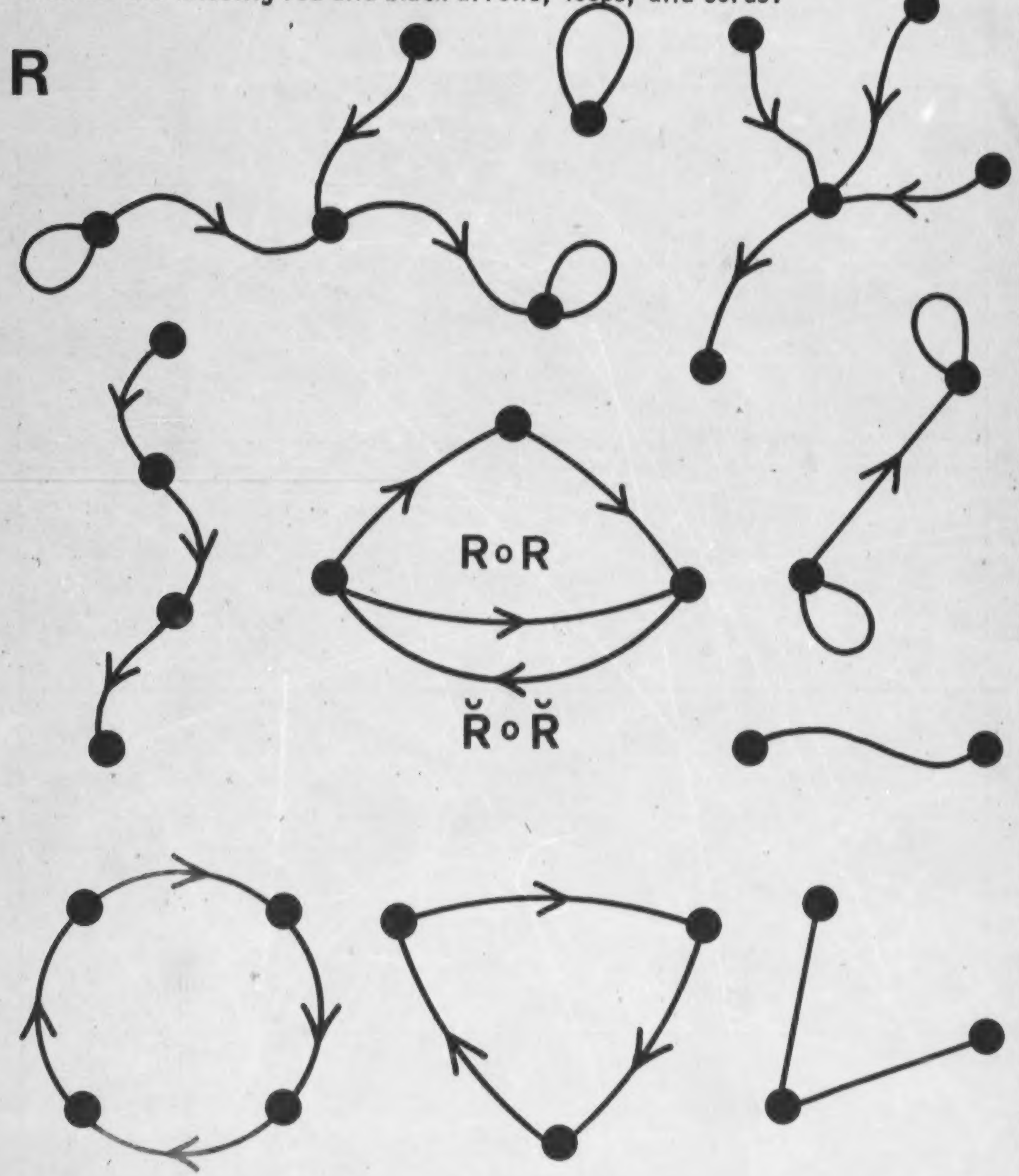
Draw all the missing red arrows, loops, and cords.



Name ______________________________

L 10 ***

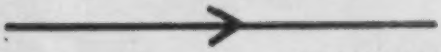
Draw all the missing red and black arrows, loops, and cords.



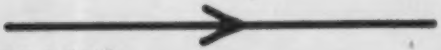
Name ______________________

L 11 *

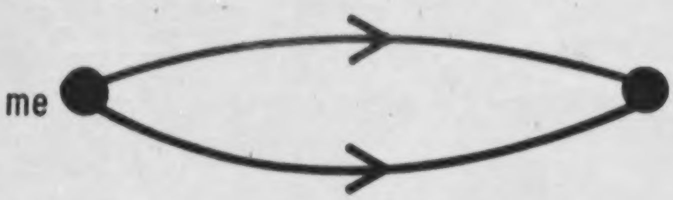
you are my father

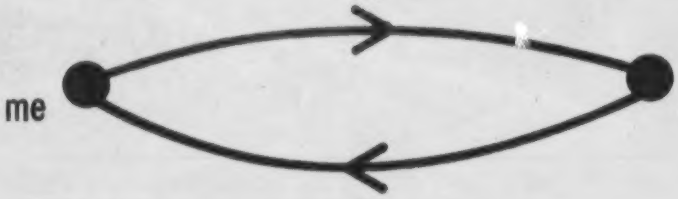


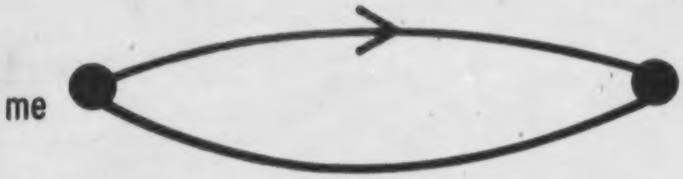
I love you



WHAT IS THE MESSAGE?

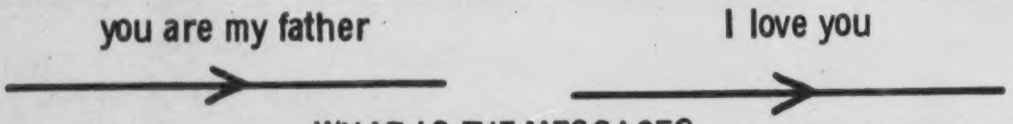




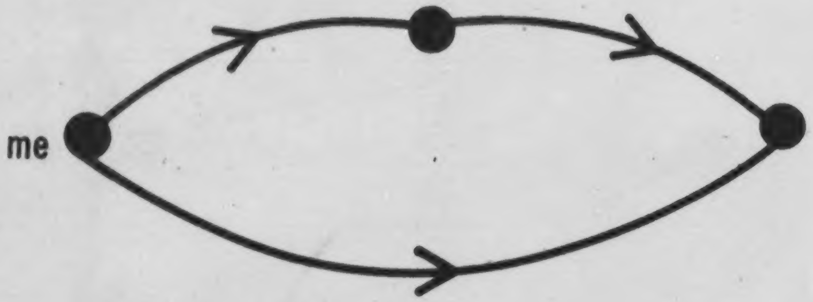


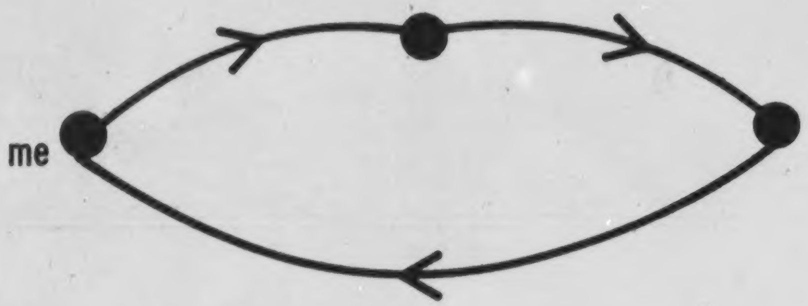
Name ____________________

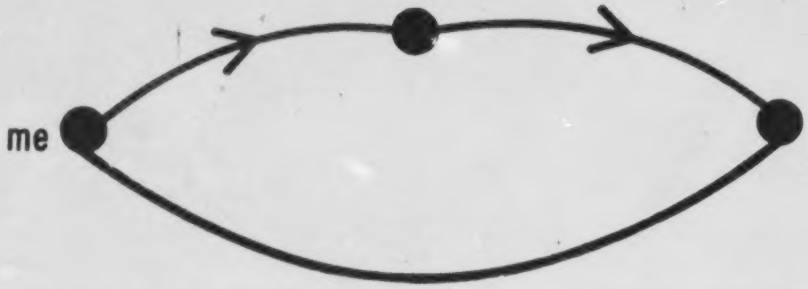
L 12 *


you are my father
I love you

WHAT IS THE MESSAGE?

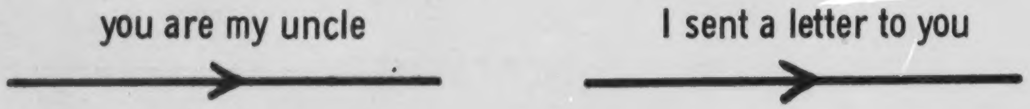




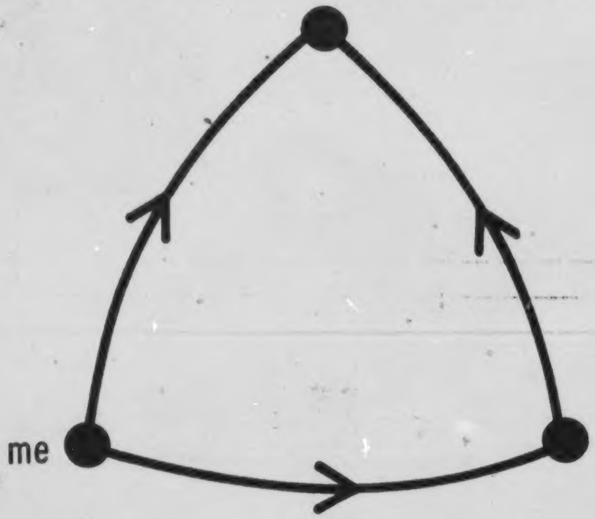


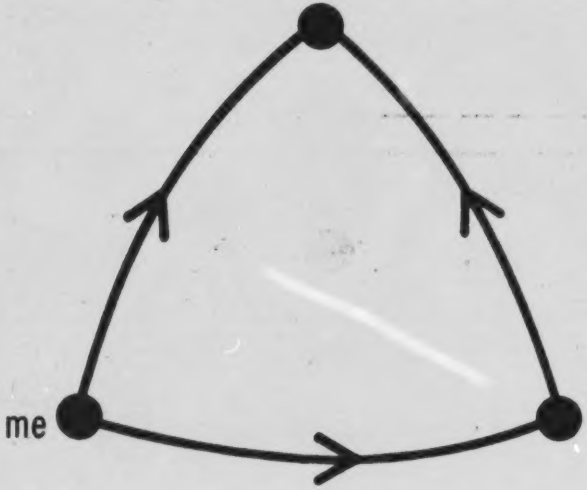
Name ______________________

L 13 *


you are my uncle
I sent a letter to you

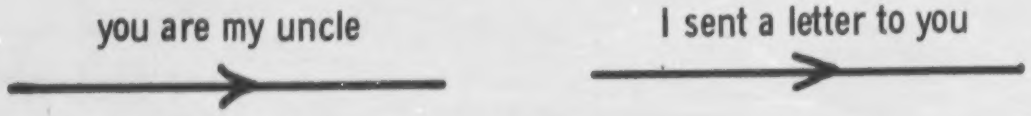
WHAT COULD THE BLACK ARROW BE FOR ?



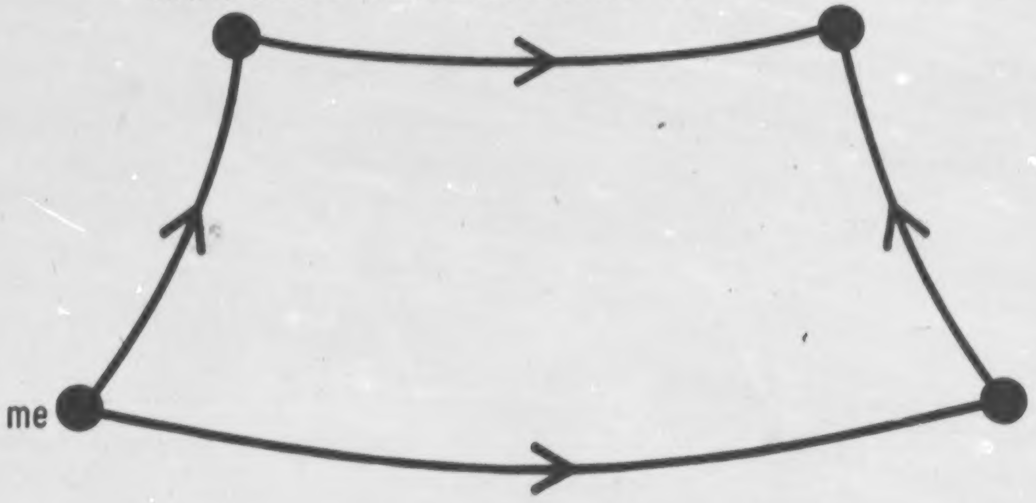


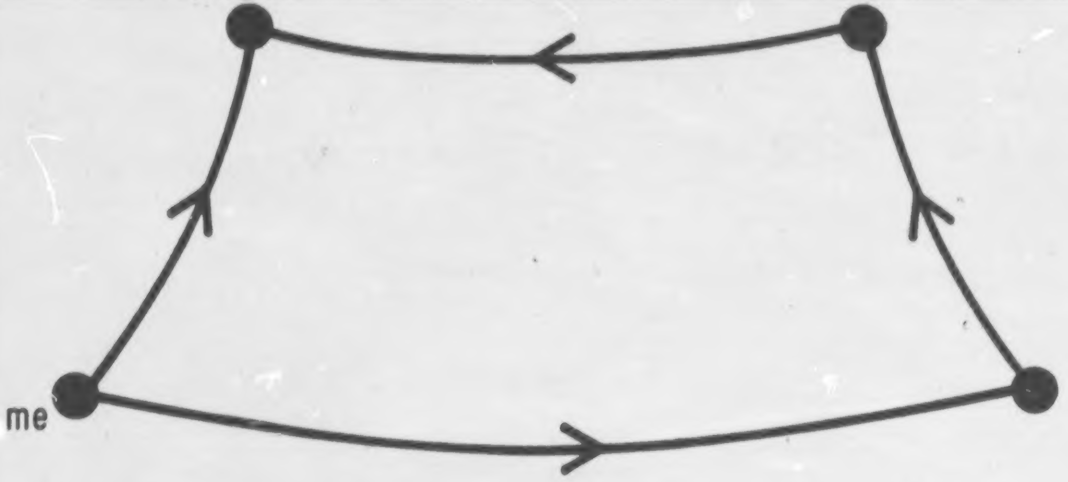
L 14 **

Name ____________________________


you are my uncle
I sent a letter to you

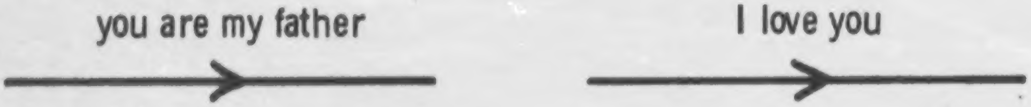
WHAT COULD THE BLACK ARROW BE FOR ?



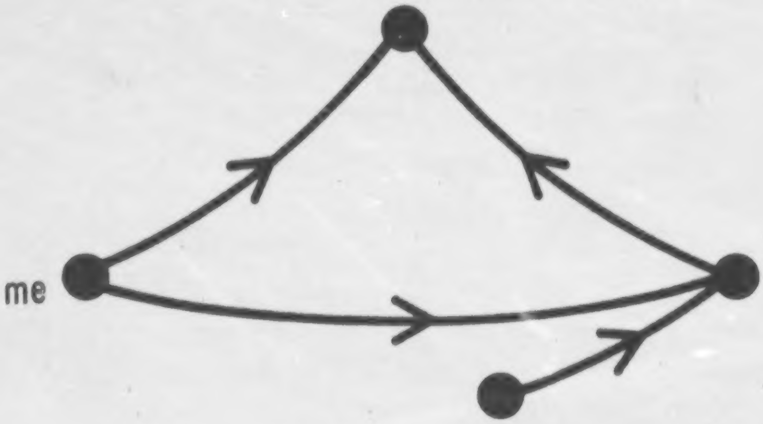


Name ________________________________

L 15 ***

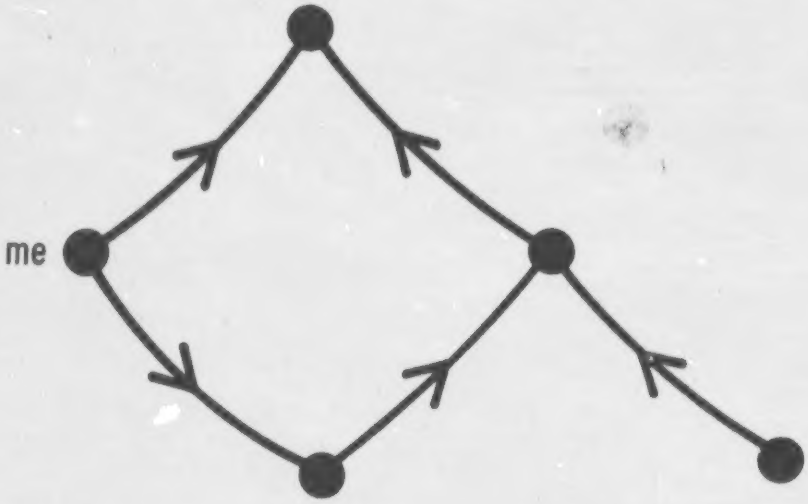

you are my father
I love you

WHAT IS THE MESSAGE ?



____

WHAT COULD THE BLACK ARROW BE FOR ?

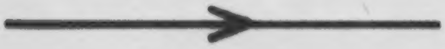


____

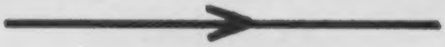
Name ____________________

L 16 ****

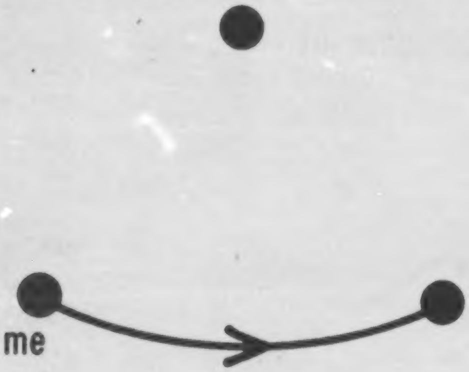
you are my uncle



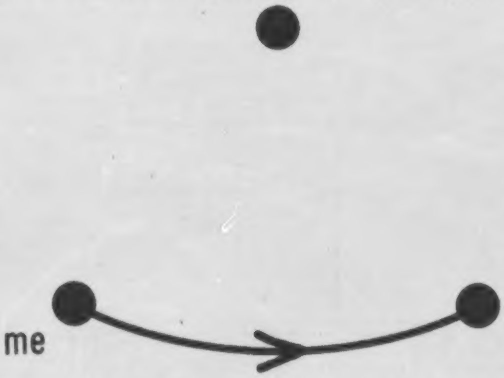
I sent a letter to you



Draw blue and red arrows so that the black arrow is for the given relation.



you sent a letter to a person
who sent a letter to me



I am an uncle of a person
who sent a letter to you

Make up your own message and draw an arrow picture for it.

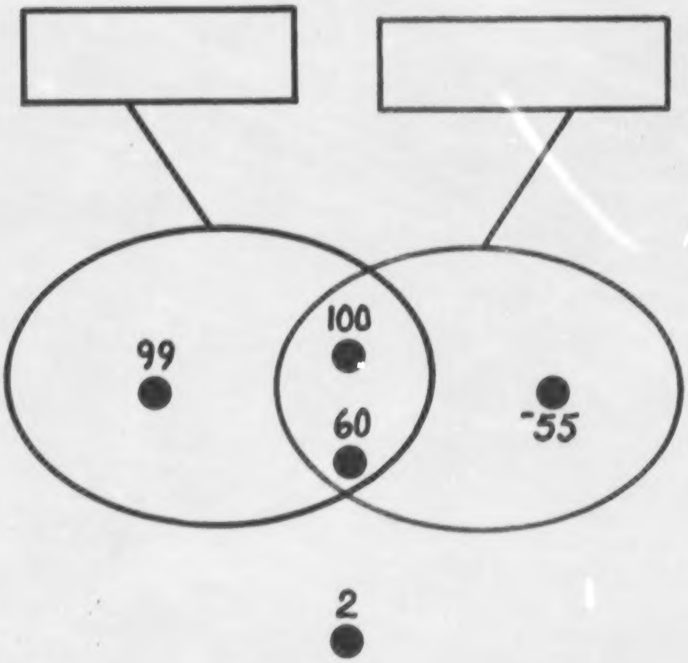
Name ______________________________

L 17 *

Use the clues in the picture to cross out labels that the strings cannot have. Some are done for you.

The RED tag is one of these:	The BLUE tag is one of these:
~~Multiples of 2~~	~~Multiples of 2~~
~~Multiples of 3~~	~~Multiples of 3~~
Multiples of 4	Multiples of 4
Multiples of 5	Multiples of 5
Multiples of 10	~~Multiples of 10~~
Odd Numbers	Odd Numbers
Positive Prime Numbers	Positive Prime Numbers
Larger than 50	Larger than 50
Smaller than 50	Smaller than 50
~~Larger than ⁻10~~	Larger than ⁻10
Smaller than ⁻10	Smaller than ⁻10
~~Positive Divisors of 12~~	~~Positive Divisors of 12~~
Positive Divisors of 18	Positive Divisors of 18
Positive Divisors of 20	Positive Divisors of 20
Positive Divisors of 24	Positive Divisors of 24
~~Positive Divisors of 27~~	~~Positive Divisors of 27~~

Complete the tags.



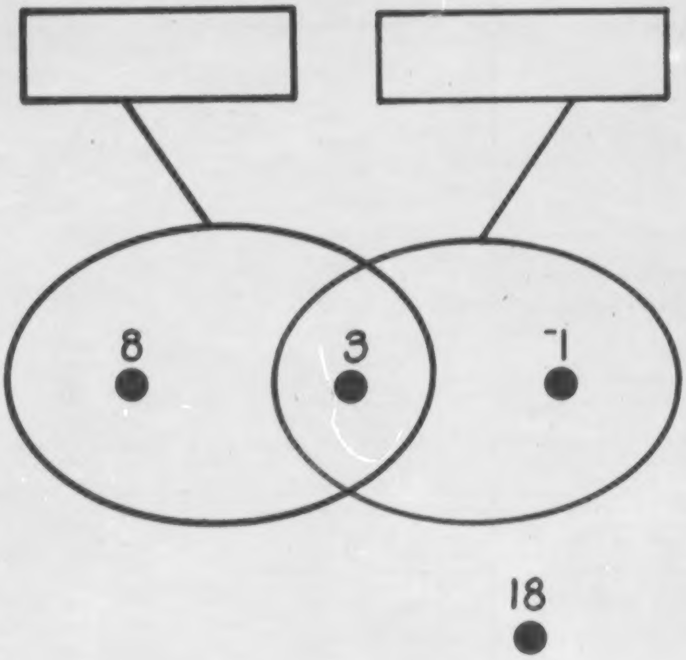
Name ______________________

L 18 **

Use the clues in the picture to cross out labels that the strings cannot have. Some are done for you.

The RED tag is one of these:	The BLUE tag is one of these:
~~Multiples of 2~~	~~Multiples of 2~~
~~Multiples of 3~~	~~Multiples of 3~~
Multiples of 4	Multiples of 4
Multiples of 5	Multiples of 5
Multiples of 10	Multiples of 10
Odd Numbers	Odd Numbers
Positive Prime Numbers	Positive Prime Numbers
Larger than 50	Larger than 50
Smaller than 50	Smaller than 50
Larger than ⁻10	Larger than ⁻10
Smaller than ⁻10	Smaller than ⁻10
Positive Divisors of 12	Positive Divisors of 12
Positive Divisors of 18	Positive Divisors of 18
Positive Divisors of 20	Positive Divisors of 20
Positive Divisors of 24	Positive Divisors of 24
Positive Divisors of 27	Positive Divisors of 27

Complete the tags.



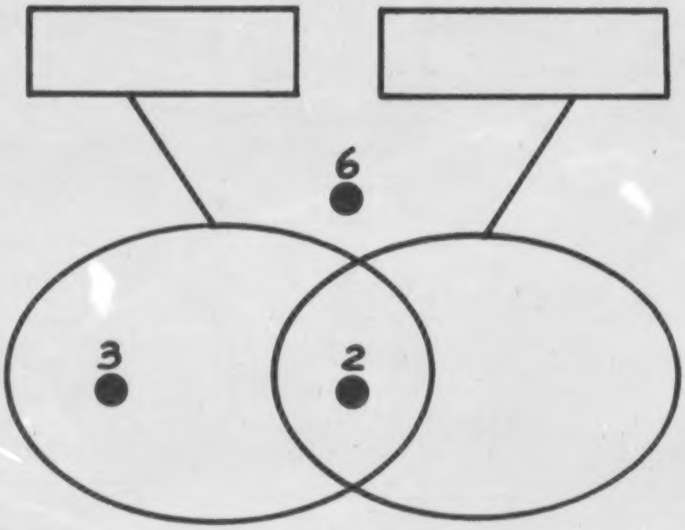
Name ______________________________

L 19 ***

Use the clues in the picture to cross out labels that the strings cannot have.

Complete the tags.

The RED tag is one of these:	The BLUE tag is one of these:
Multiples of 2	Multiples of 2
Multiples of 3	Multiples of 3
Multiples of 4	Multiples of 4
Multiples of 5	Multiples of 5
Multiples of 10	Multiples of 10
Odd Numbers	Odd Numbers
Positive Prime Numbers	Positive Prime Numbers
Larger than 50	Larger than 50
Smaller than 50	Smaller than 50
Larger than ⁻10	Larger than ⁻10
Smaller than ⁻10	Smaller than ⁻10
Positive Divisors of 12	Positive Divisors of 12
Positive Divisors of 18	Positive Divisors of 18
Positive Divisors of 20	Positive Divisors of 20
Positive Divisors of 24	Positive Divisors of 24
Positive Divisors of 27	Positive Divisors of 27



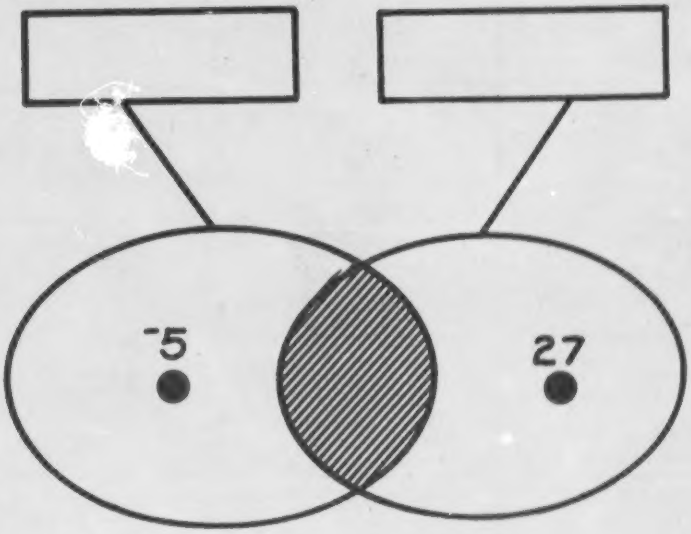
Name ______________________

L 20 ****

Use the clues in the picture to cross out labels that the strings cannot have. The hatcning is a clue.

The RED tag is one of these:	The BLUE tag is one of these:
Multiples of 2	Multiples of 2
Multiples of 3	Multiples of 3
Multiples of 4	Multiples of 4
Multiples of 5	Multiples of 5
Multiples of 10	Multiples of 10
Odd Numbers	Odd Numbers
Positive Prime Numbers	Positive Prime Numbers
Larger than 50	Larger than 50
Smaller than 50	Smaller than 50
Larger than $^{-}10$	Larger than $^{-}10$
Smaller than $^{-}10$	Smaller than $^{-}10$
Positive Divisors of 12	Positive Divisors of 12
Positive Divisors of 18	Positive Divisors of 18
Positive Divisors of 20	Positive Divisors of 20
Positive Divisors of 24	Positive Divisors of 24
Positive Divisors of 27	Positive Divisors of 27

Complete the tags.

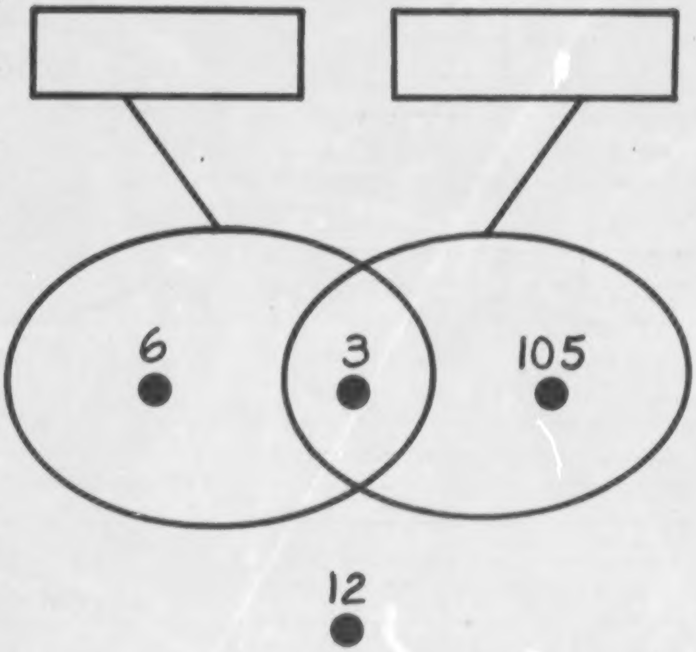


Name ______________________________ **L 21** *

Cross out the labels that the strings cannot have.
Some are done for you.

The RED tag is one of these:	The BLUE tag is one of these:
~~Multiples of 2~~	Multiples of 2
~~Multiples of 3~~	~~Multiples of 3~~
Multiples of 4	Multiples of 4
Multiples of 5	~~Multiples of 5~~
Multiples of 10	Multiples of 10
~~Odd Numbers~~	Odd Numbers
Positive Prime Numbers	~~Positive Prime Numbers~~
Larger than 50	Larger than 50
~~Smaller than 50~~	Smaller than 50
Larger than ¯10	~~Larger than ¯10~~
Smaller than ¯10	~~Smaller than ¯10~~
Positive Divisors of 12	~~Positive Divisors of 12~~
Positive Divisors of 18	Positive Divisors of 18
Positive Divisors of 20	Positive Divisors of 20
~~Positive Divisors of 24~~	Positive Divisors of 24
~~Positive Divisors of 27~~	Positive Divisors of 27

Complete the tags.



Put these numbers in the string picture.

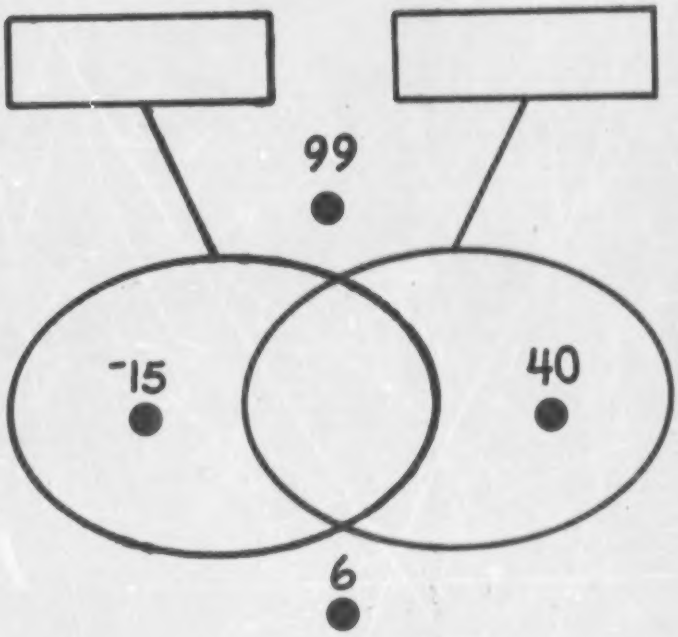
¯1; 1; 2; 4; 5; 9; 18; 24; 100; ¯100.

L 22 **

Name ______________________________

Cross out the labels that the strings cannot have.
Some are done for you.

The RED tag is one of these:	The BLUE tag is one of these:
~~Multiples of 2~~	~~Multiples of 2~~
~~Multiples of 3~~	~~Multiples of 3~~
Multiples of 4	Multiples of 4
Multiples of 5	Multiples of 5
Multiples of 10	Multiples of 10
Odd Numbers	Odd Numbers
~~Positive Prime Numbers~~	~~Positive Prime Numbers~~
~~Larger than 50~~	~~Larger than 50~~
Smaller than 50	Smaller than 50
Larger than $^{-}10$	Larger than $^{-}10$
Smaller than $^{-}10$	Smaller than $^{-}10$
~~Positive Divisors of 12~~	~~Positive Divisors of 12~~
Positive Divisors of 18	Positive Divisors of 18
Positive Divisors of 20	Positive Divisors of 20
Positive Divisors of 24	Positive Divisors of 24
Positive Divisors of 27	Positive Divisors of 27



The red tag is ______________________.

The blue tag could be

______________ or ______________.

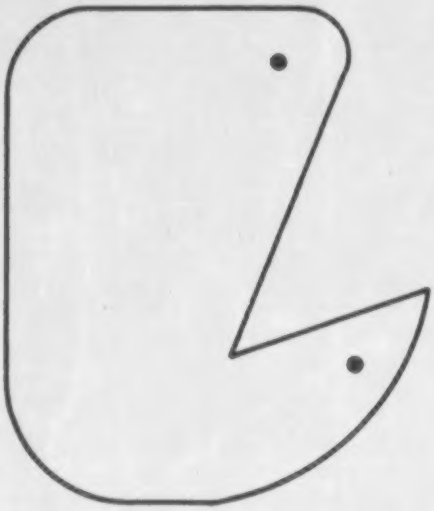
Exactly two of the following numbers cannot be put in the string picture because the blue string is still not known. Circle these two numbers. Put all the others in the string picture.

$^{-}100$; $^{-}80$; $^{-}55$; $^{-}5$; 0; 7; 8; 10; 20; 60.

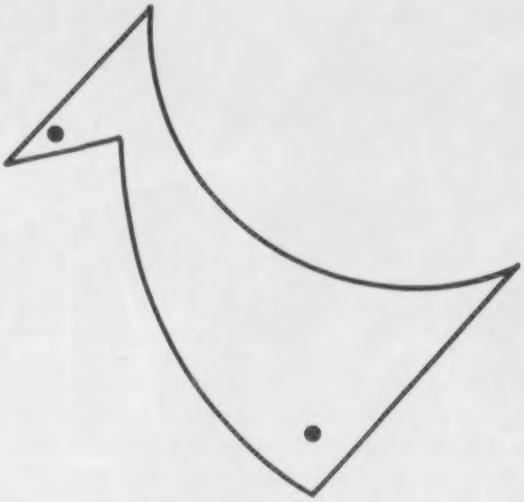
G 01 *

Name ______________________________

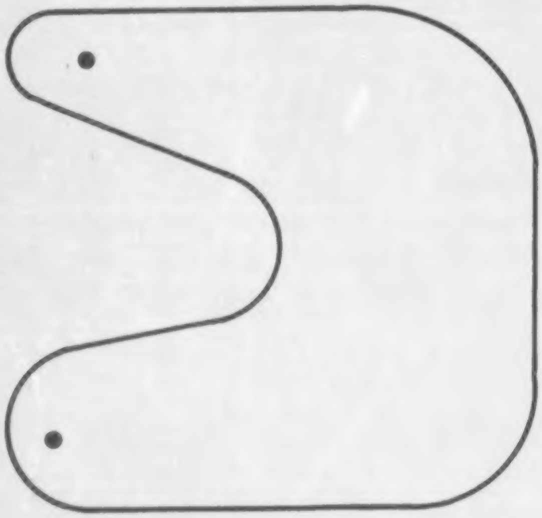
In each picture connect the dots with a zigzag. Do not go out of the school yard. Measure the length of each zigzag. Try to make your zigzag as short as possible.



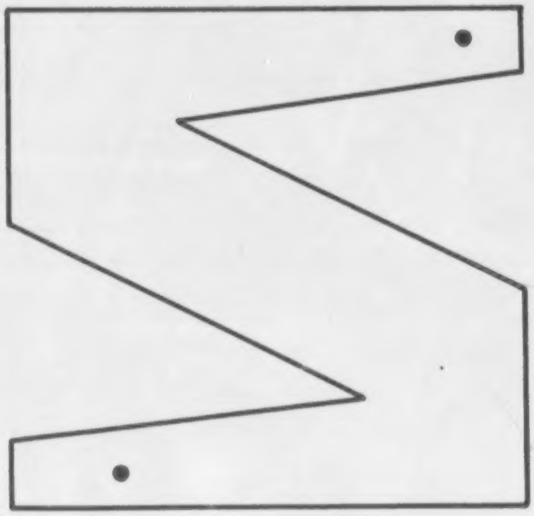
length of zigzag = __________cm



length of zigzag = __________cm



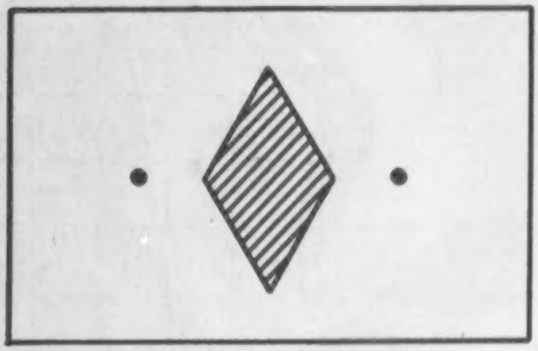
length of zigzag = __________cm



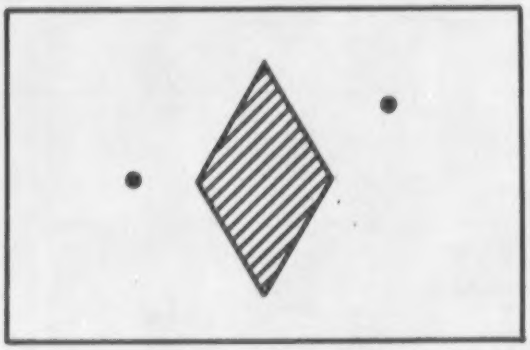
length of zigzag = __________cm

Name ______________________________ **G 02 ****

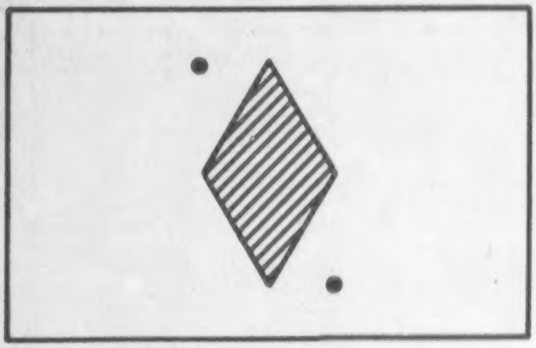
In each picture connect the dots with a zigzag. Do not go out of the school yard or through the building. Measure the length of each zigzag. Try to draw a zigzag which makes the number story true.



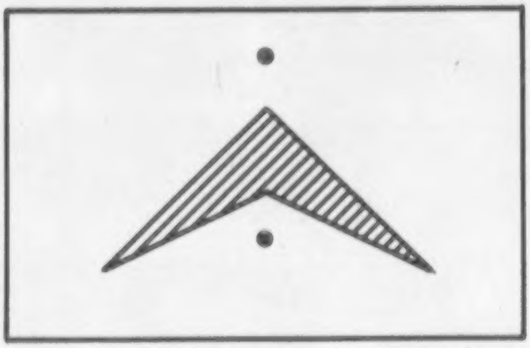
_________cm is less than 5.8 cm.



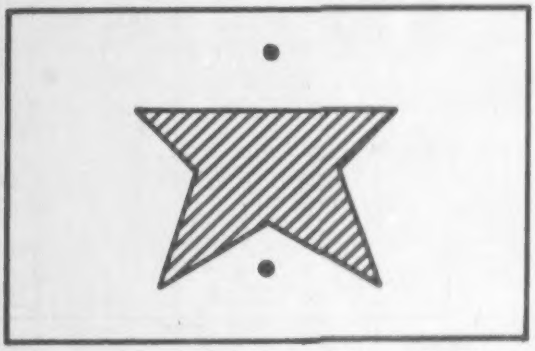
_________cm is less than 5.2 cm.



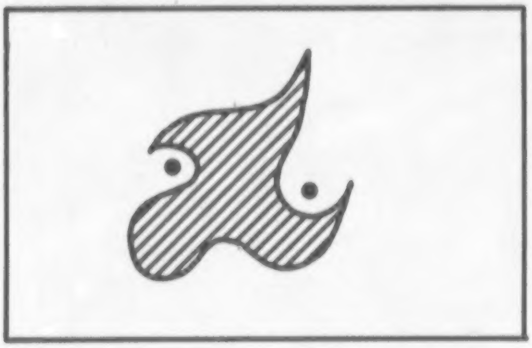
_________cm is less than 4.9 cm.



_________cm is less than 7.2 cm.



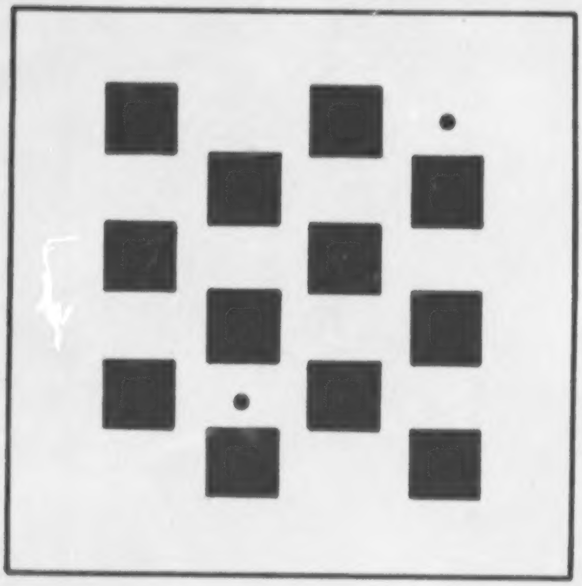
_________cm is less than 6.9 cm.



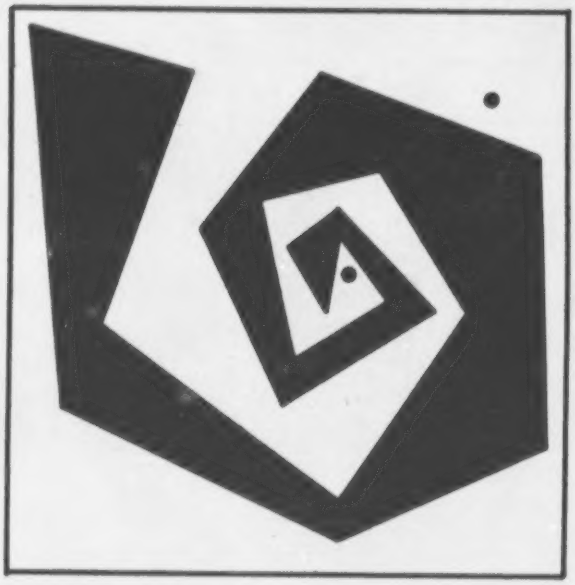
_________cm is less than 6.7 cm.

G 03 ***

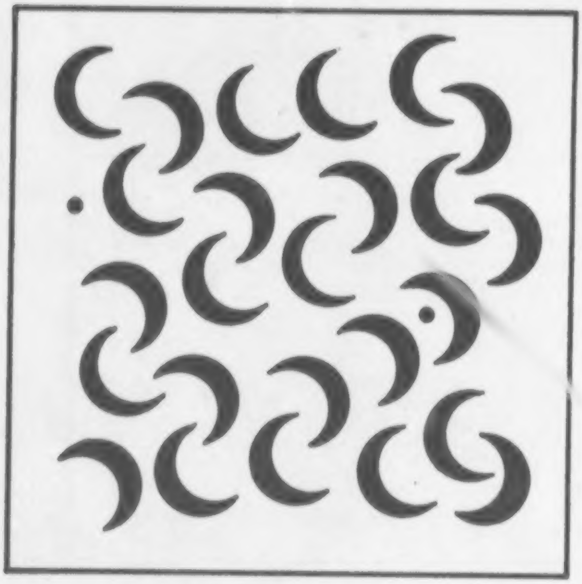
In each picture connect the dots with a zigzag. Do not go out of the school yard or through the shaded areas. Make your zigzags as short as possible.



length = ________ cm



length = ________ cm



length = ________ cm

Make your own problem here.

length = ________ cm

Name ______________________

G 04 *

In each grid draw a red shape with the correct area. Rule: the corners of your shapes must be dots of the grid.

area = 3 cm²

area = 4 cm²

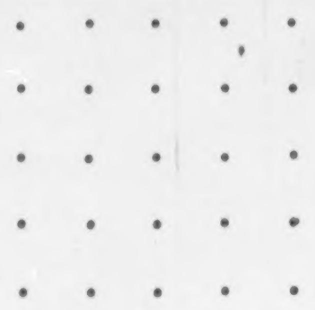
area = 5 cm²

area = 6 cm²

area = 7 cm²

area = 8 cm²

area = 9 cm²

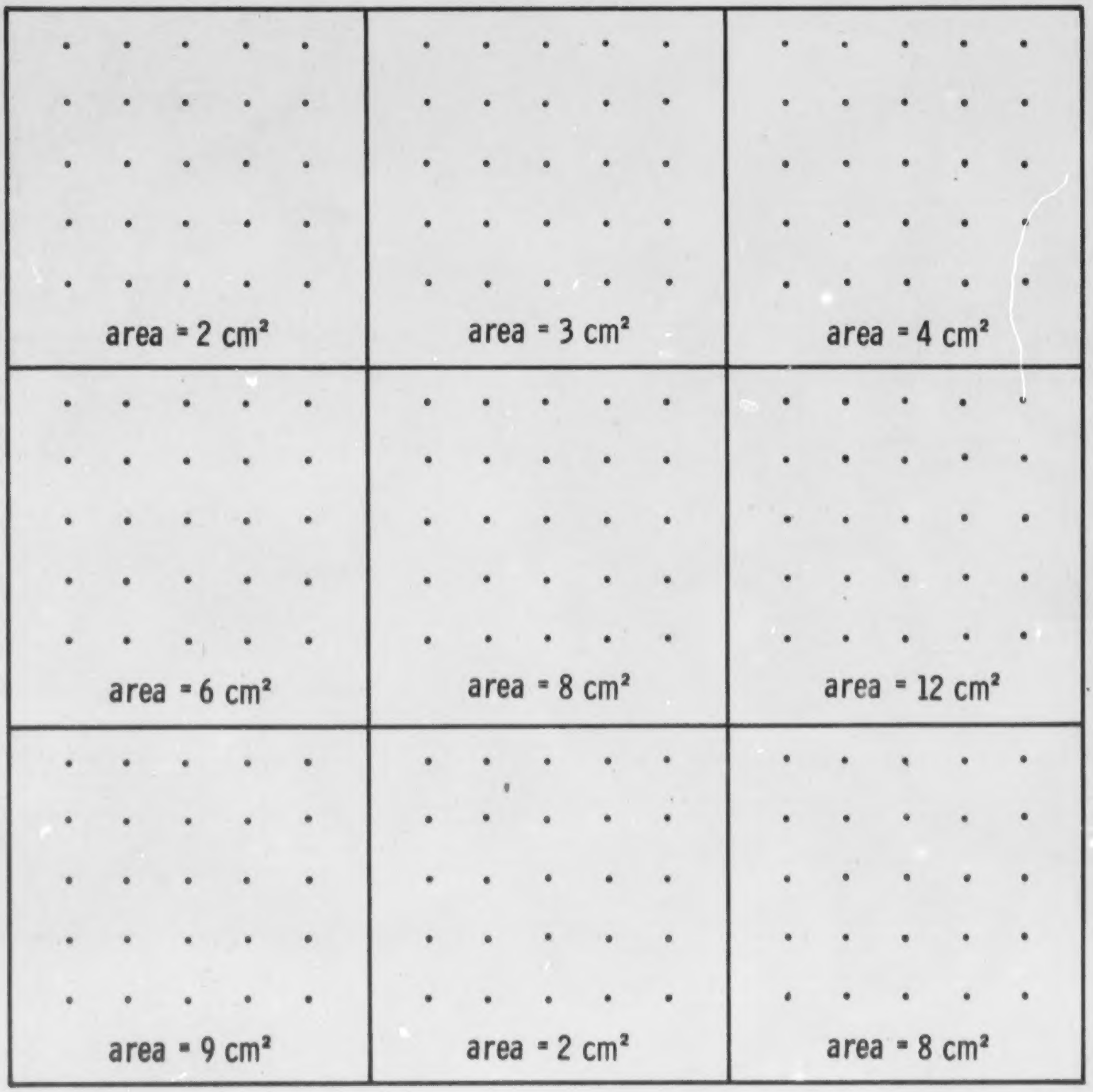

area = 12 cm²

area = 16 cm²

G 05 **

Name ______________________________

In each grid draw a blue square or a blue rectangle with the correct area.


area = $2\ \text{cm}^2$
area = $3\ \text{cm}^2$
area = $4\ \text{cm}^2$
area = $6\ \text{cm}^2$
area = $8\ \text{cm}^2$
area = $12\ \text{cm}^2$
area = $9\ \text{cm}^2$
area = $2\ \text{cm}^2$
area = $8\ \text{cm}^2$

Try to make squares with the last three areas.

Name ______________________________

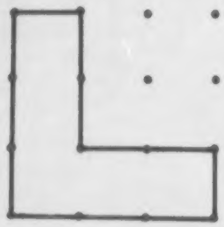
G06 ***

Draw as many red shapes as you can with area 5 cm². The perimeter of a shape is the length of its border. Find and write the perimeter of each of your shapes next to it. Circle the one with the smallest perimeter.

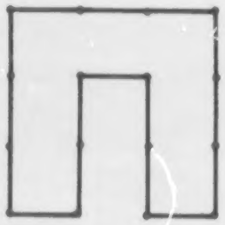
Name ______________________

G 07 *

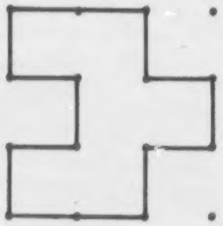
Find the area of each shape.



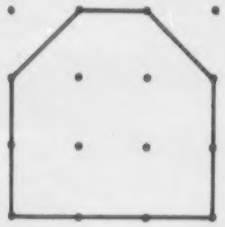
area = ______ cm^2



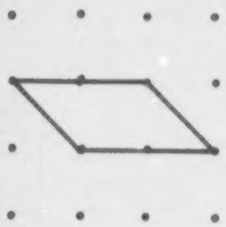
area = ______ cm^2



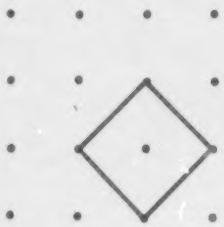
area = ______ cm^2



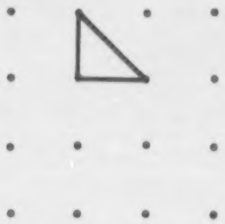
area = ______ cm^2



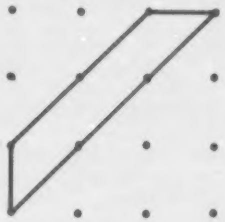
area = ______ cm^2



area = ______ cm^2



area = ______ cm^2



area = ______ cm^2

n each grid below draw a red rectangle with the correct area.

area = 8 cm^2

area = 6 cm^2

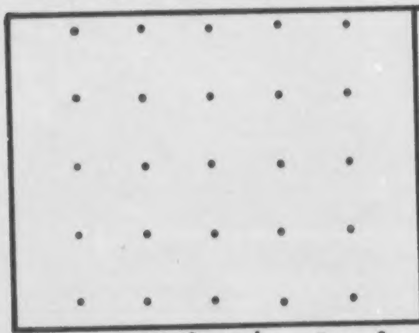
area = 12 cm^2

Now draw a biue triangle in each rectangle. Make the area of each triangle half of the rectangle's area.

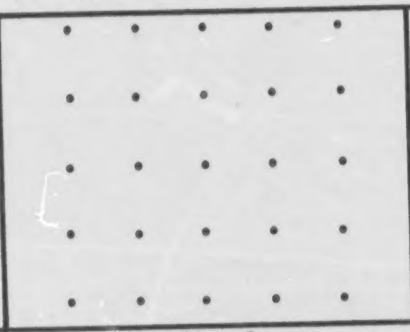
G 08 **

Name ____________________

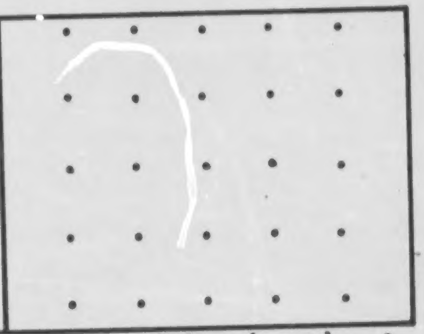
In each grid draw a blue triangle with the correct area. You may draw red rectangles to help find the triangles.



area of triangle = 1 cm²

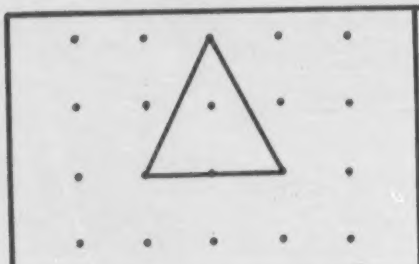


area of triangle = 2 cm²

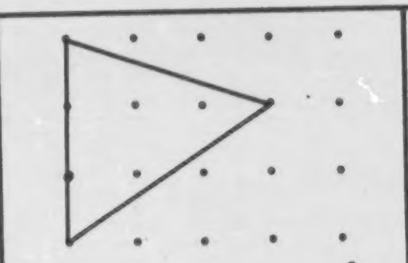


area of triangle = $1\frac{1}{2}$ cm²

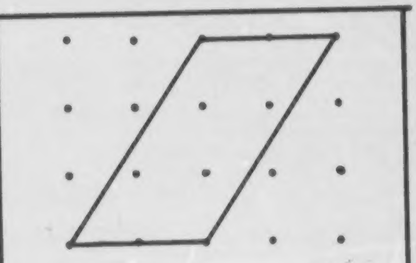
Find the area of each shape.



area = ________ cm²

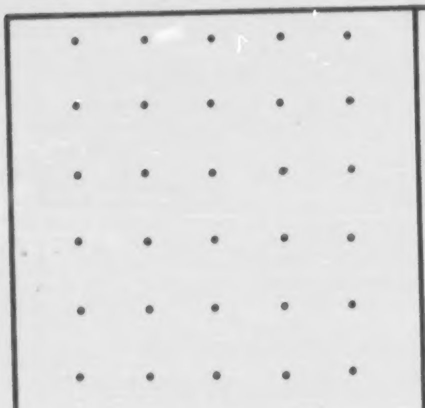


area = ________ cm²

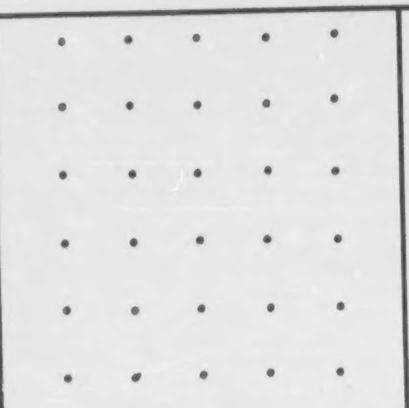


area = ________ cm²

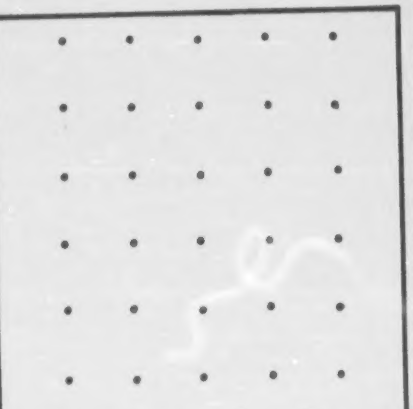
In each grid draw a blue triangle with the correct area.



area = 8 cm²



area = 5 cm²



area = $4\frac{1}{2}$ cm²

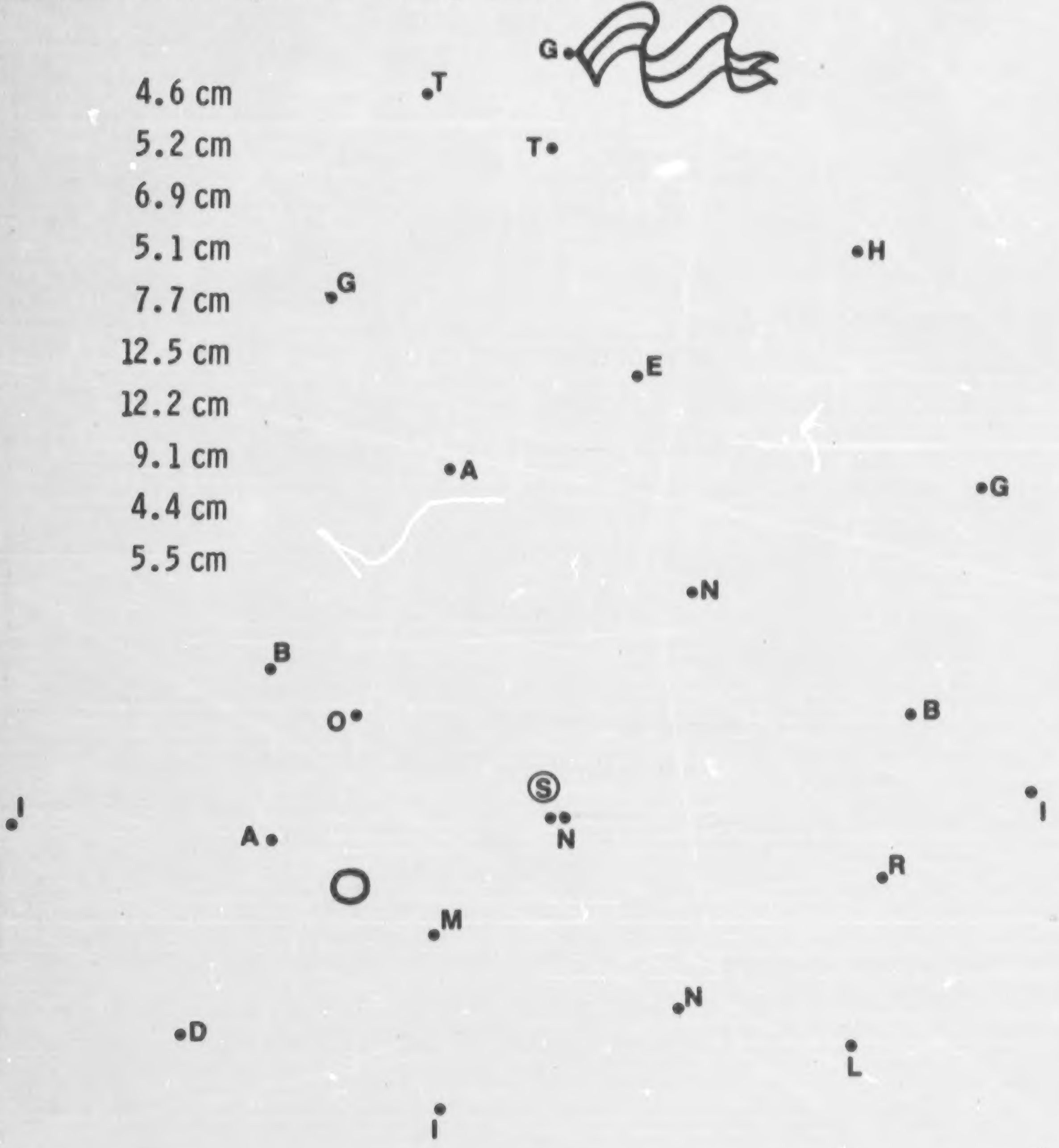
Name ______________________________

G 09 *

Follow the dots. Draw a zigzag starting at the dot marked Ⓢ.

Use this list of lengths for the segments of the zigzag.

4.6 cm
5.2 cm
6.9 cm
5.1 cm
7.7 cm
12.5 cm
12.2 cm
9.1 cm
4.4 cm
5.5 cm

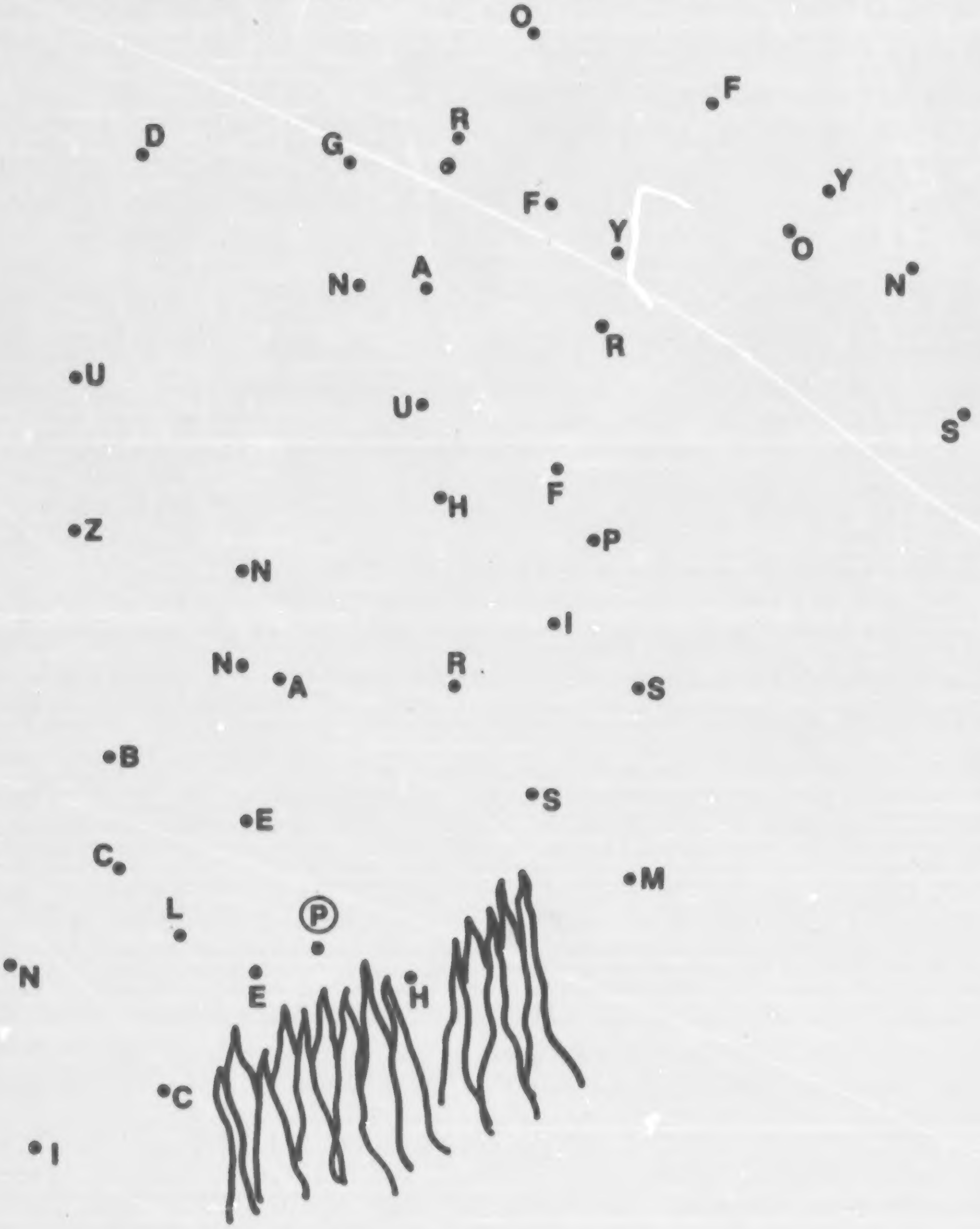

G
T
T
H
G
E
A
G
N
B
O
B
S
I
A
N
I
O
R
M
N
D
L
I

S_______ ____

G 10 **

Name ______________________________

Follow the dots. Draw a zigzag starting at the dot marked Ⓟ.
Use this list of lengths for the segments of the zigzag.


O
F
R
D
G
Y
F
Y
O
A
N
N
R
U
U
S
F
H
Z
P
N
I
N
A
R
S
B
S
E
C
M
L
Ⓟ
N
E
H
C
I

1.0 cm
1.2 cm
3.8 cm
4.3 cm
3.7 cm
0.6 cm
3.9 cm
1.4 cm
2.0 cm
1.8 cm
1.7 cm
5.5 cm
4.1 cm
3.5 cm
3.1 cm
2.7 cm
3.3 cm
2.3 cm
2.6 cm
3.3 cm

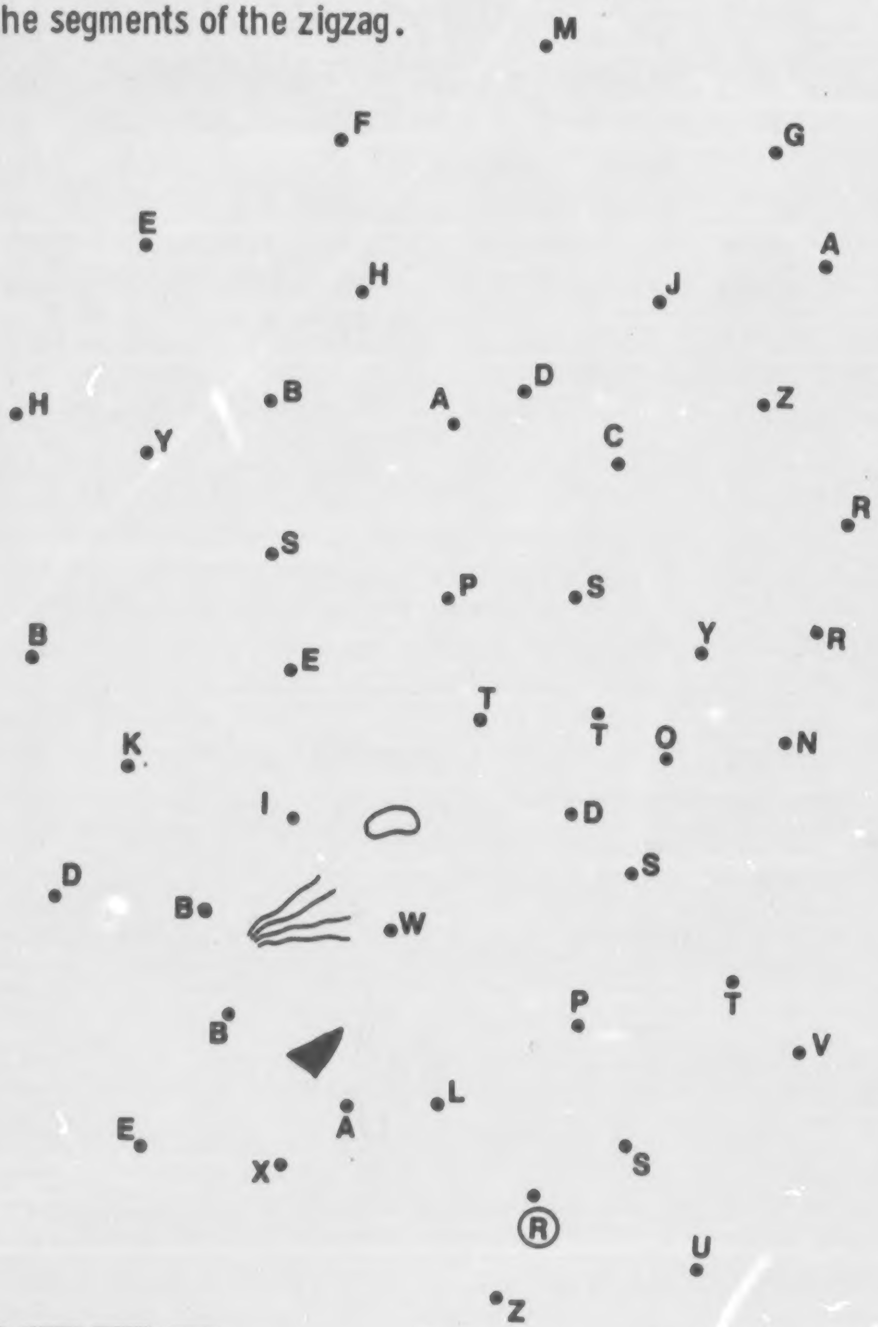
P_______ ______
___ _ ____

Name ________________________________

G 11 ***

Follow the dots. Draw a zigzag starting at the dot marked Ⓡ. Use this list of lengths for the segments of the zigzag.

3.1 cm
2.2 cm
1.6 cm
1.9 cm
3.1 cm
3.9 cm
4.9 cm
5.2 cm
4.8 cm
3.7 cm
4.2 cm
3.8 cm
1.7 cm
2.9 cm
3.4 cm
2.9 cm


M
F
G
E
A
H
J
H
B
A
D
Z
Y
C
R
S
P
S
B
Y
R
E
T
T
O
N
K
I
D
D
S
B
W
P
T
B
V
L
E
A
X
S
Ⓡ
U
Z

R _ _ _ _ _ _

_ _ _

_ _ _ _ _ _ _

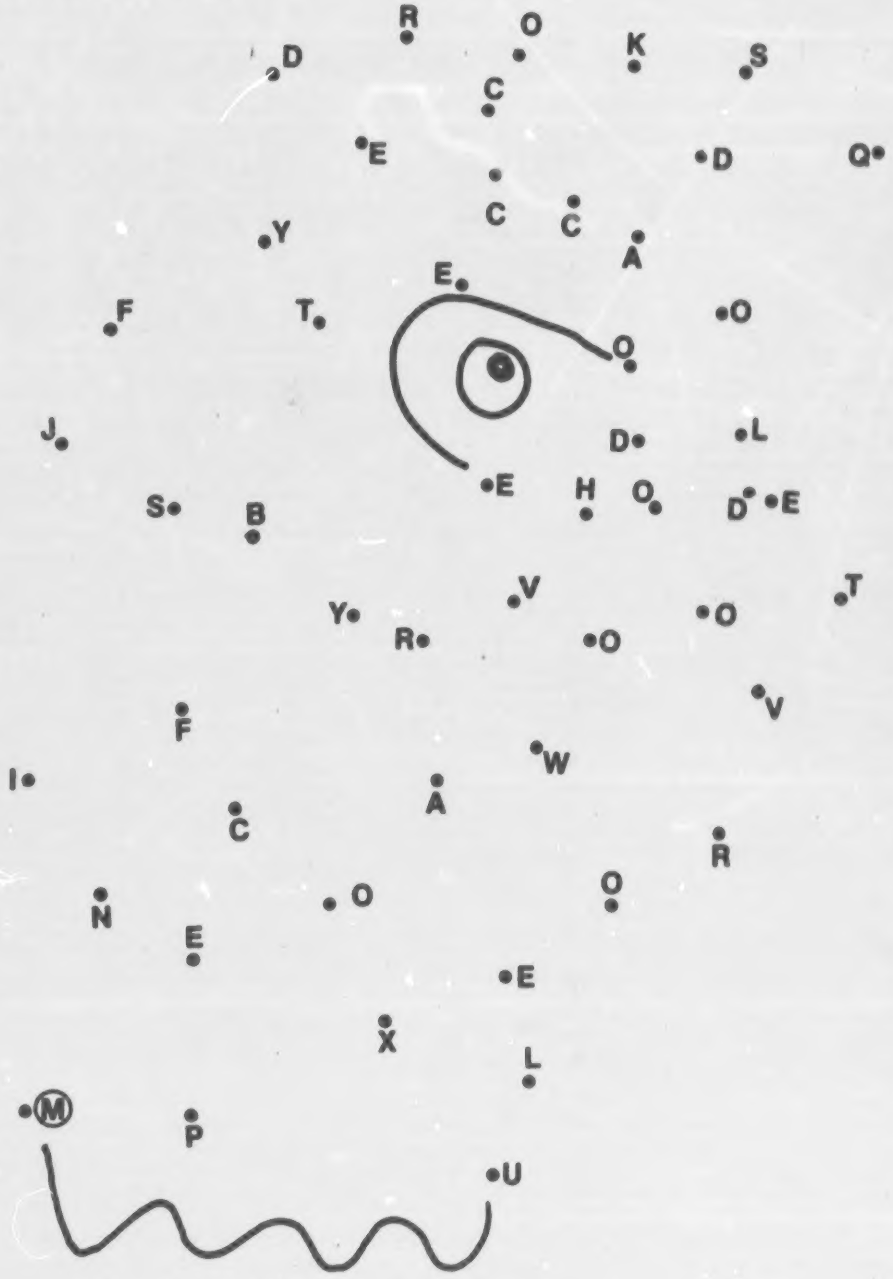
G 12 **

Name ______________________

Follow the dots. Draw a zigzag starting at the dot marked Ⓜ.

Use this list of lengths for the segments of the zigzag.

4.9 cm
4.5 cm
3.5 cm
2.2 cm
3.7 cm
2.4 cm
1.8 cm
2.3 cm
2.2 cm
2.5 cm
1.5 cm
2.3 cm
1.6 cm
1.1 cm
1.5 cm
1.1 cm
0.3 cm
1.4 cm
1.7 cm
2.2 cm
1.9 cm
1.8 cm
1.6 cm
2.1 cm
2.5 cm
2.8 cm
4.2 cm
4.7 cm


R
O
D
K
S
C
E
D
Q
C
C
Y
A
E
F
T
O
O
J
D
L
E
H
O
D
E
S
B
V
T
Y
O
R
O
V
F
W
I
A
C
R
N
O
O
E
E
X
L
Ⓜ
P
U

M______ _____-_-_______ ___

___ ___ ___

Name ______________________

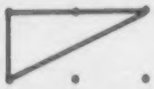
G 13 *

Draw a red rectangle or square with each of these areas:

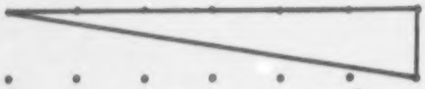
1 cm², 3 cm², 4 cm², 6 cm², 7 cm², 8 cm², 9 cm², 10 cm².

Write the area of each shape next to it.

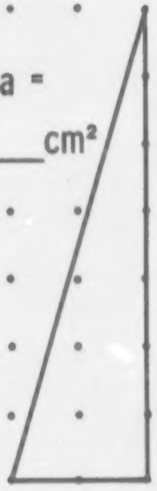
Find the area of each of these triangles.



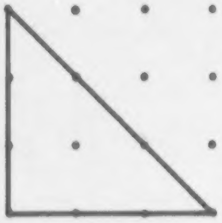
area = _____cm²



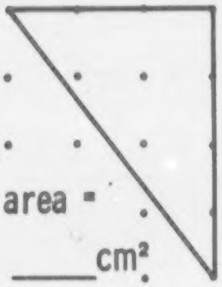
area = _____cm²



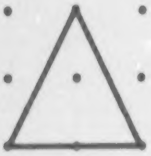
area = _____cm²



area = _____cm²



area = _____cm²

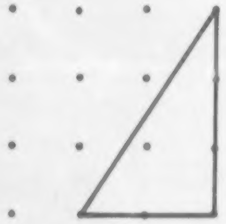


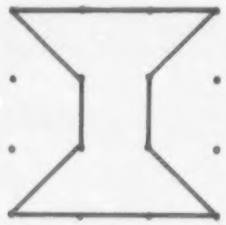
area = _____cm²

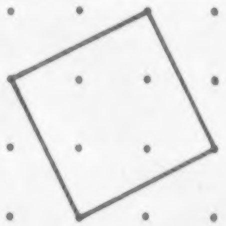
Name ______________________

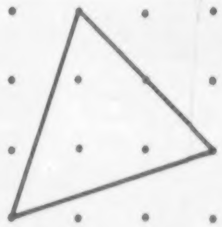
G 14 **

Find the area of each shape.


area = ____ cm²


area = ____ cm²


area = ____ cm²


area = ____ cm²

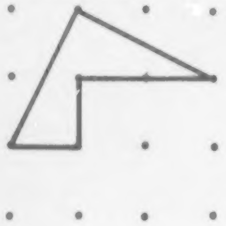
In each grid draw a red shape with the correct area.

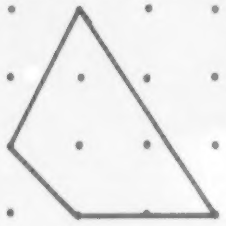
area = 7½ cm²

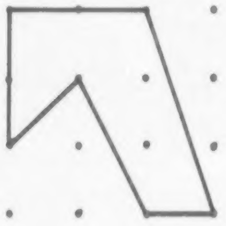
area = 9½ cm²

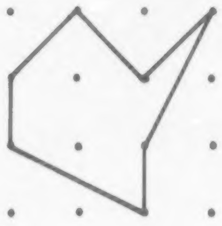
area = 5½ cm²

Find the area of each shape.


area = ____ cm²


area = ____ cm²

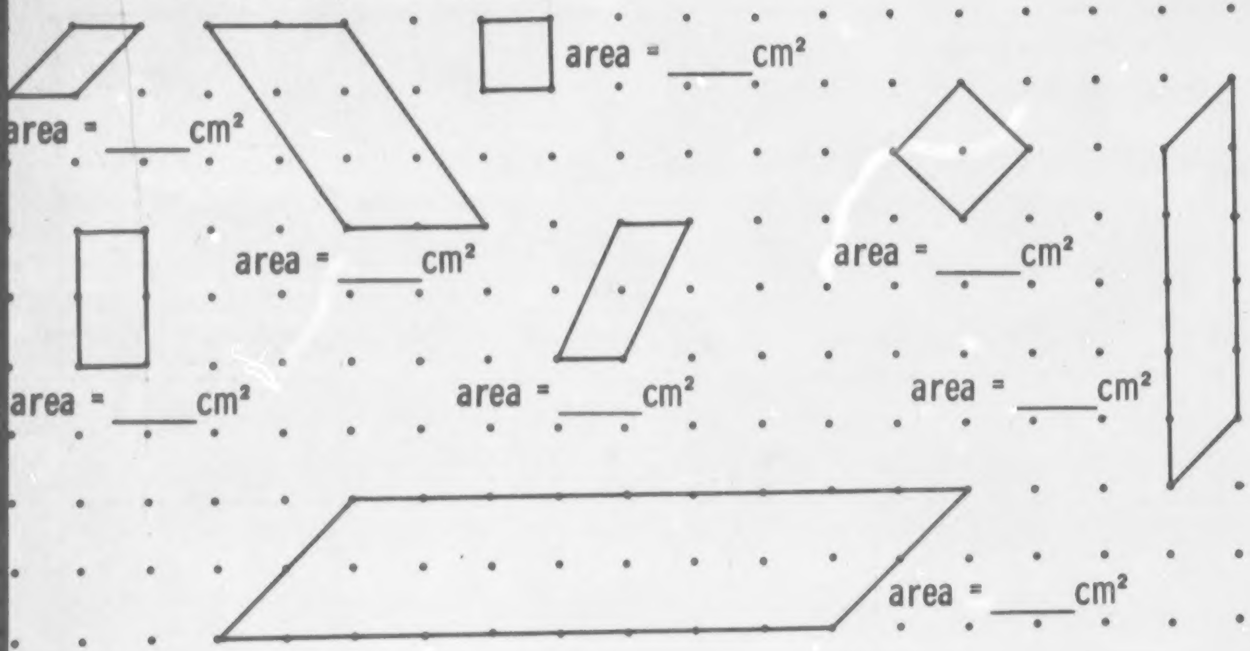

area = ____ cm²


area = ____ cm²

G 15 *

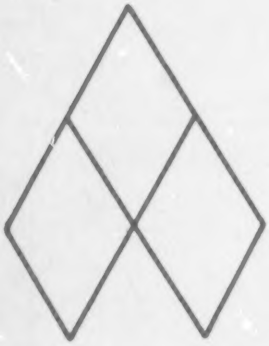
Name ______________________

Find the area of each parallelogram.



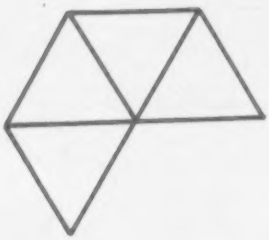
area = ____cm²
area = ____cm²
area = ____cm²
area = ____cm²
area = ____cm²
area = ____cm²
area = ____cm²
area = ____cm²

How many parallelograms can you find in each of these pictures? Trace each one that you find with a different color.



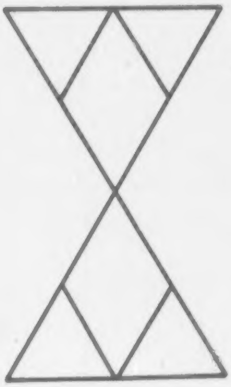
How many parallelograms?

______



How many parallelograms?

______

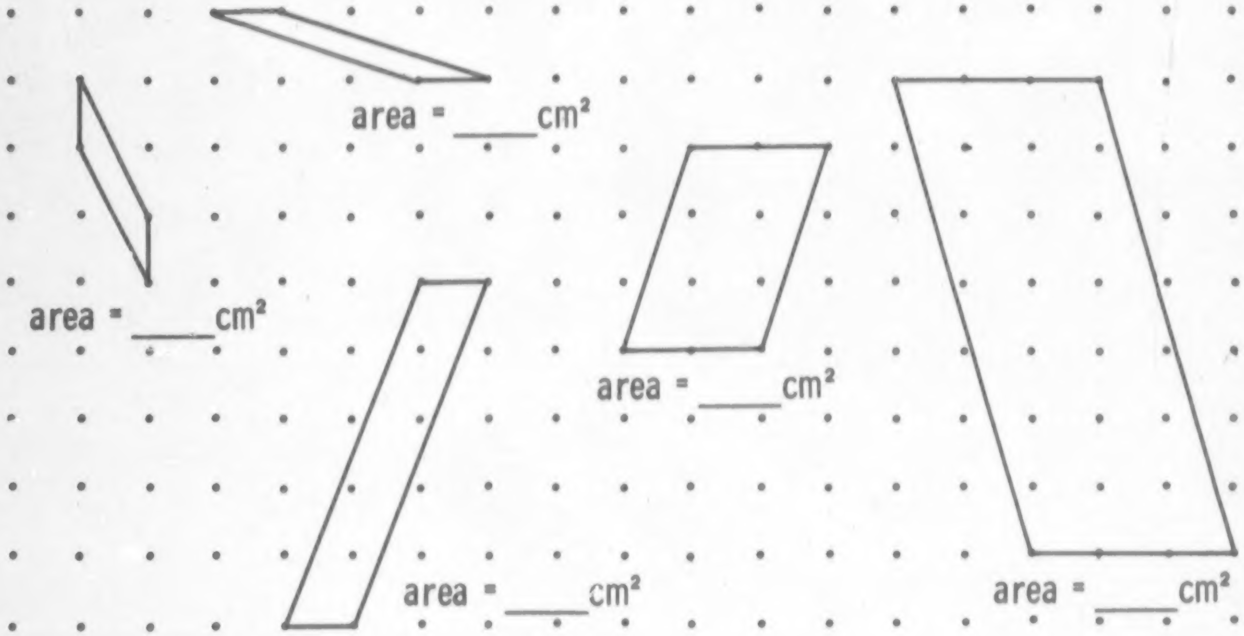


How many parallelograms?

______

Name ______________________ **G 16 ****

Find the area of each parallelogram.



area = ____cm²

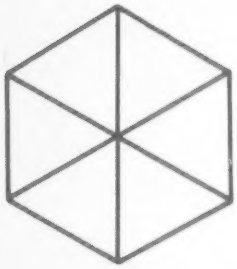
area = ____cm²

area = ____cm²

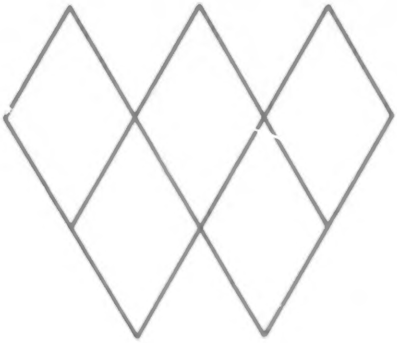
area = ____cm²

area = ____cm²

How many parallelograms can you find in each of these pictures? Trace each one that you find in a different color.



How many parallelograms? _____



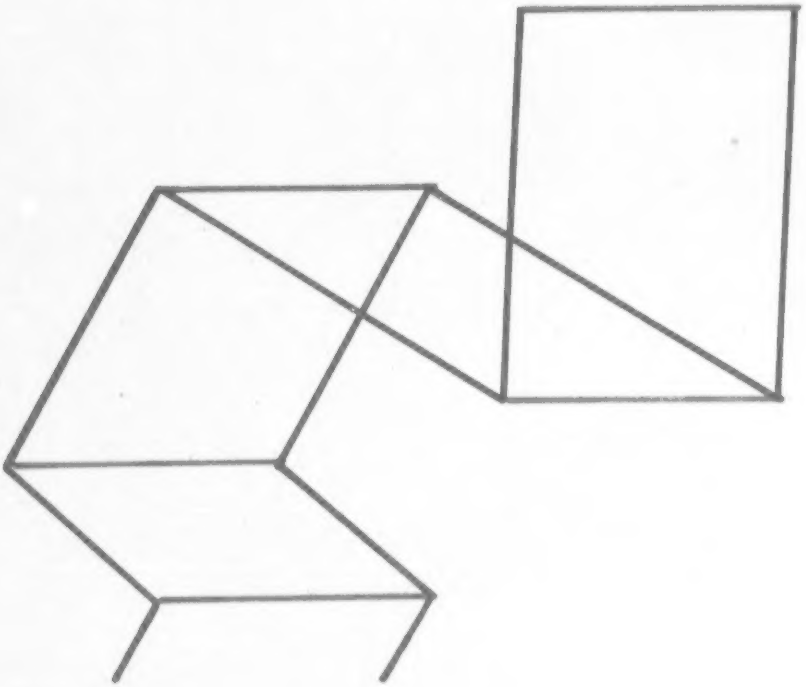
How many parallelograms? _____

Name ______________________________

G 17 *

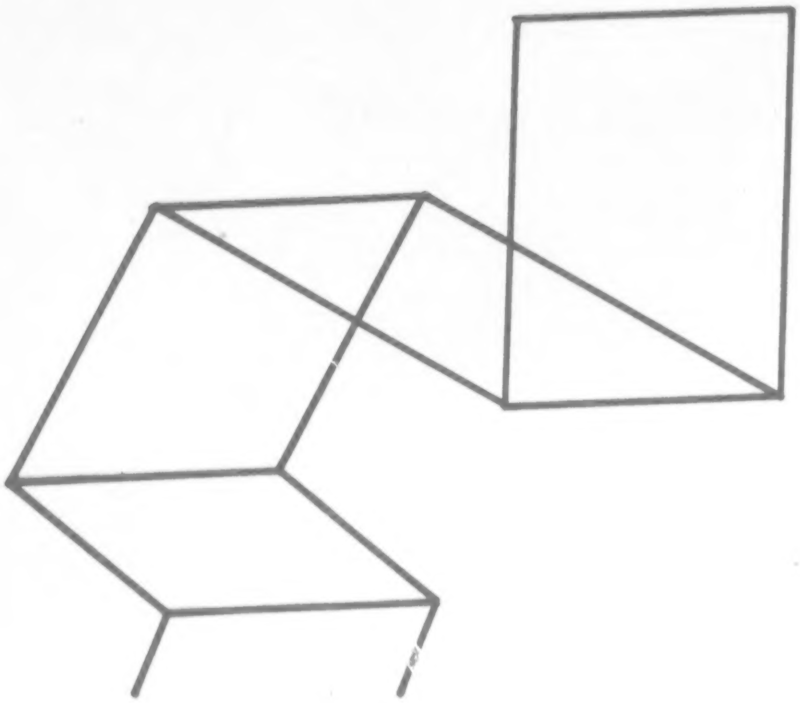
These parallelograms join the red line segments like the cars of a train.

Use your translator to add at least six more red line segments and parallelograms to the train.



These parallelograms join the red line segments like the cars of a train.

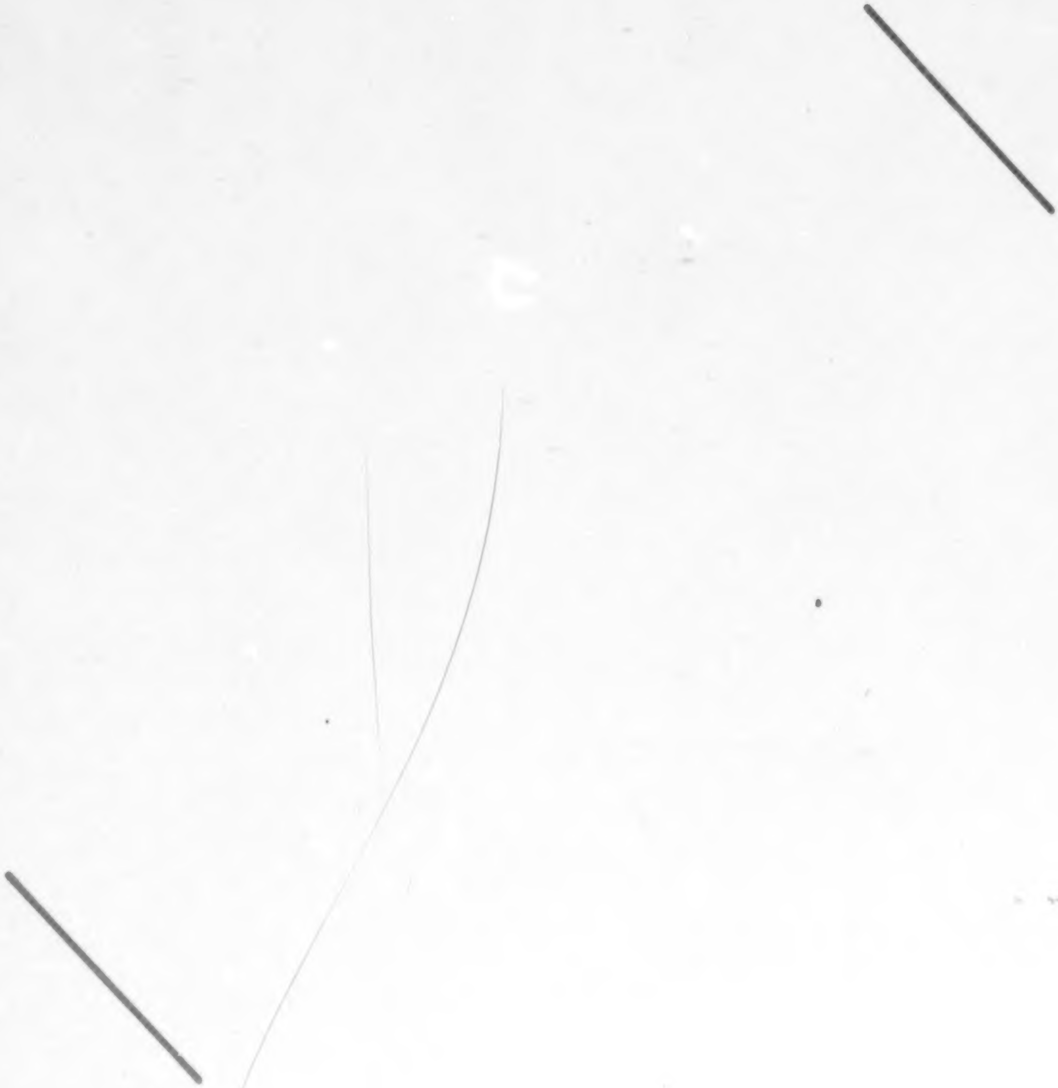
- Use your translator to add at least six more red line segments and parallelograms to the train.



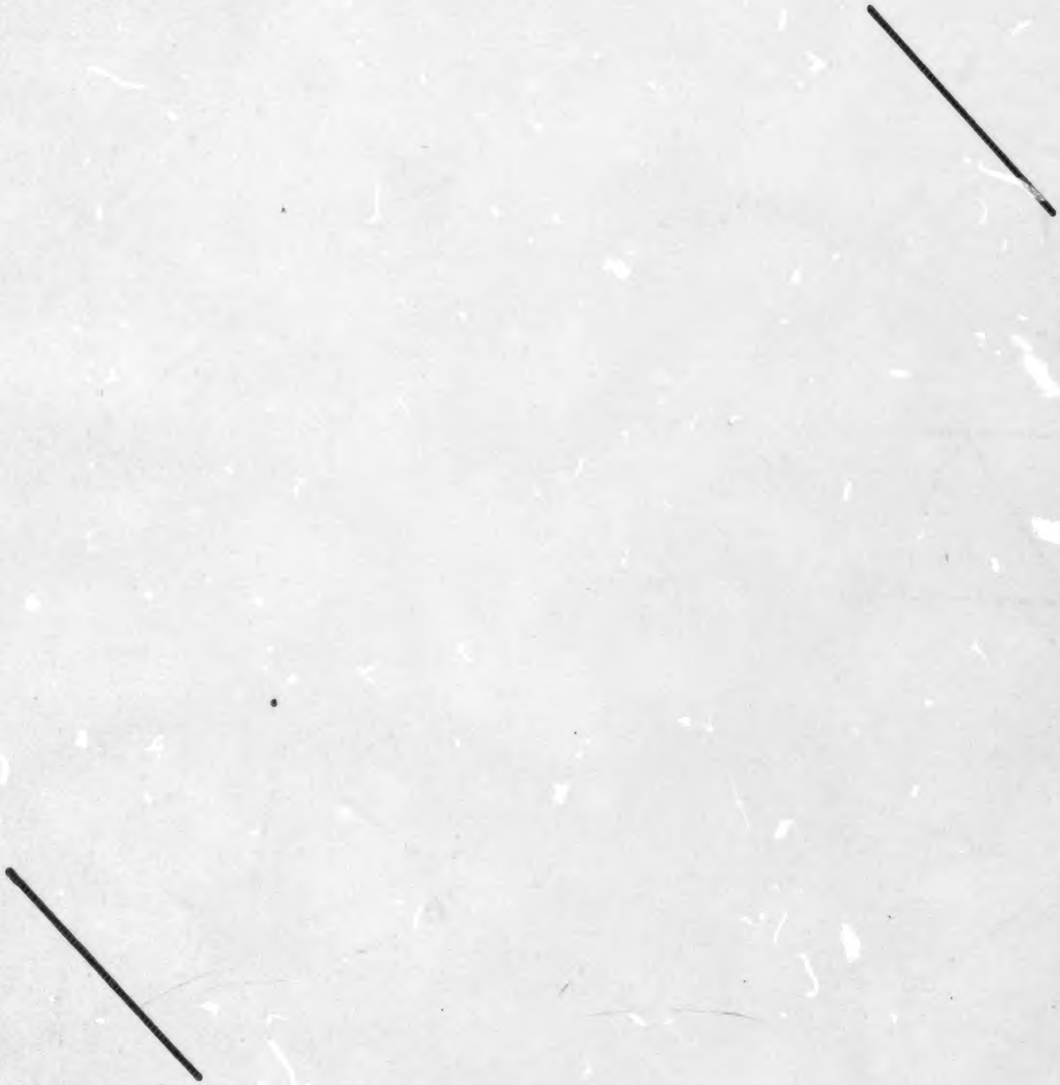
Name ______________________

G 18 **

Draw a train with at least four parallelograms connecting the two red line segments.



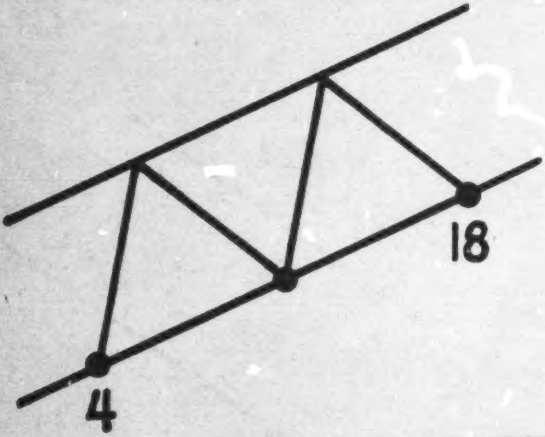
Draw a train with at least four parallelograms connecting the two red line segments.

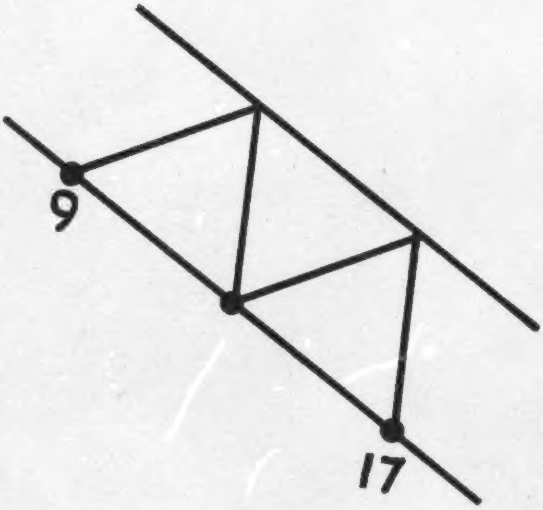


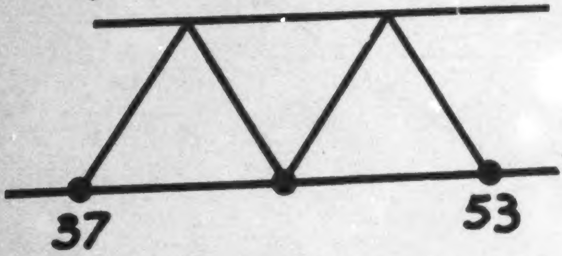
Name ______________________________

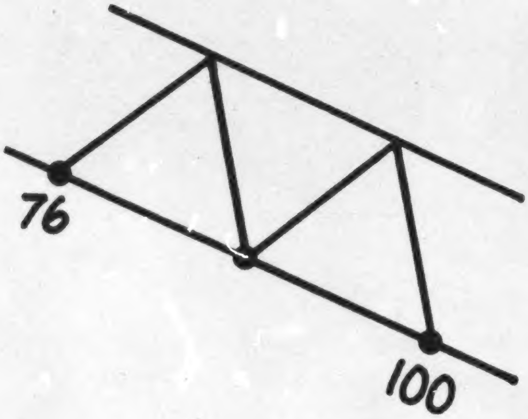
G 19 *

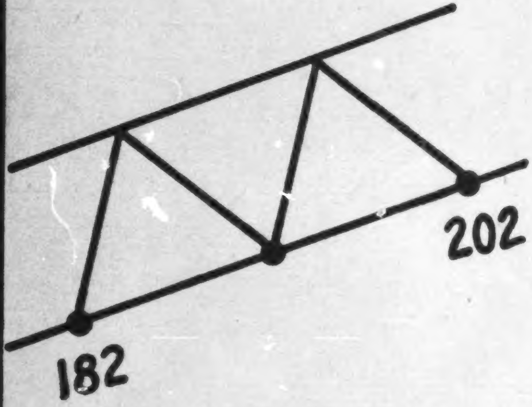
Label the dots.

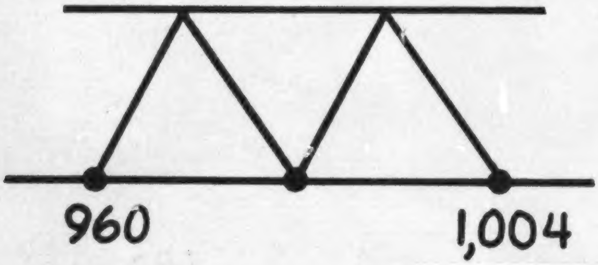








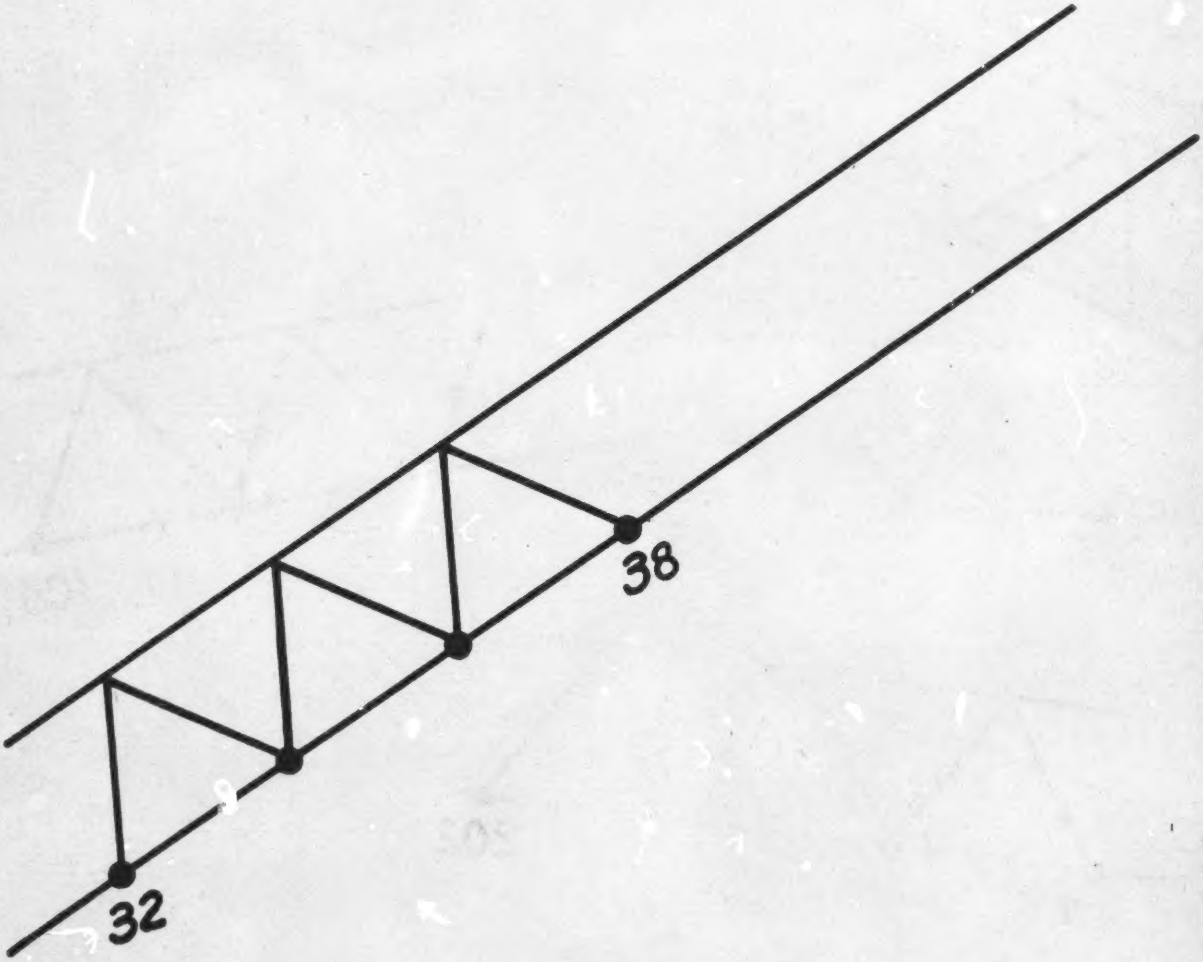




G 20 **

Name ____________________

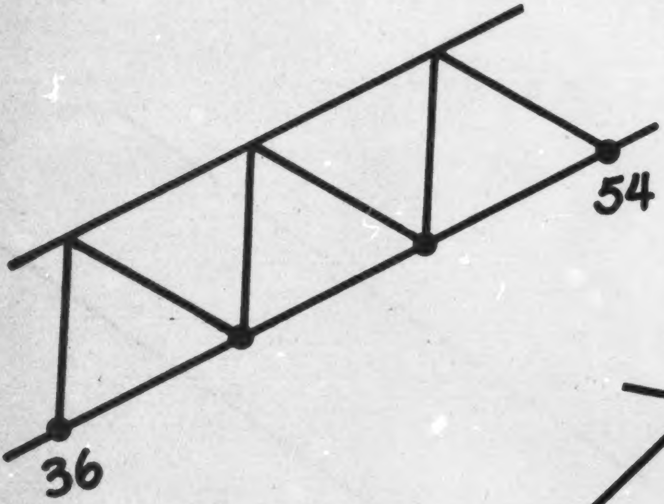
Label the dots. Use your translator to place a dot correctly for 44.

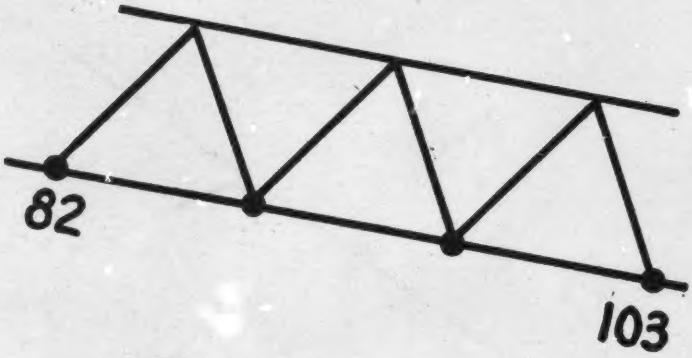


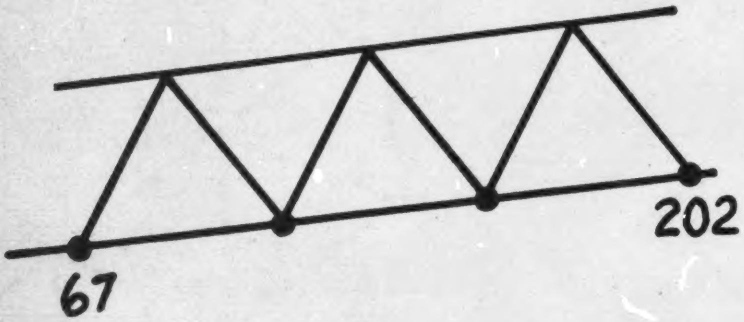
Name ______________________________

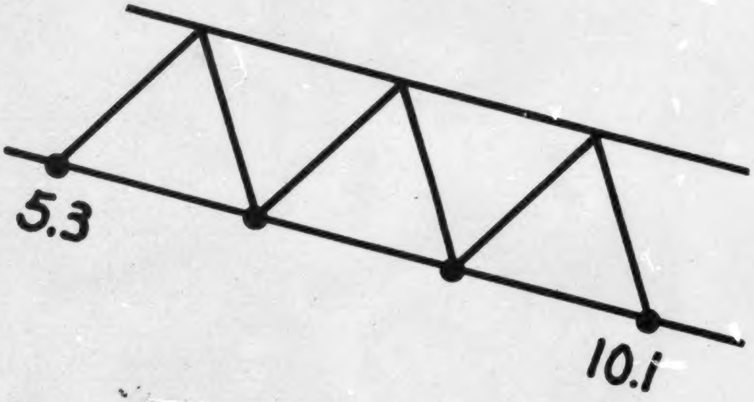
G 21 ***

Label the dots.





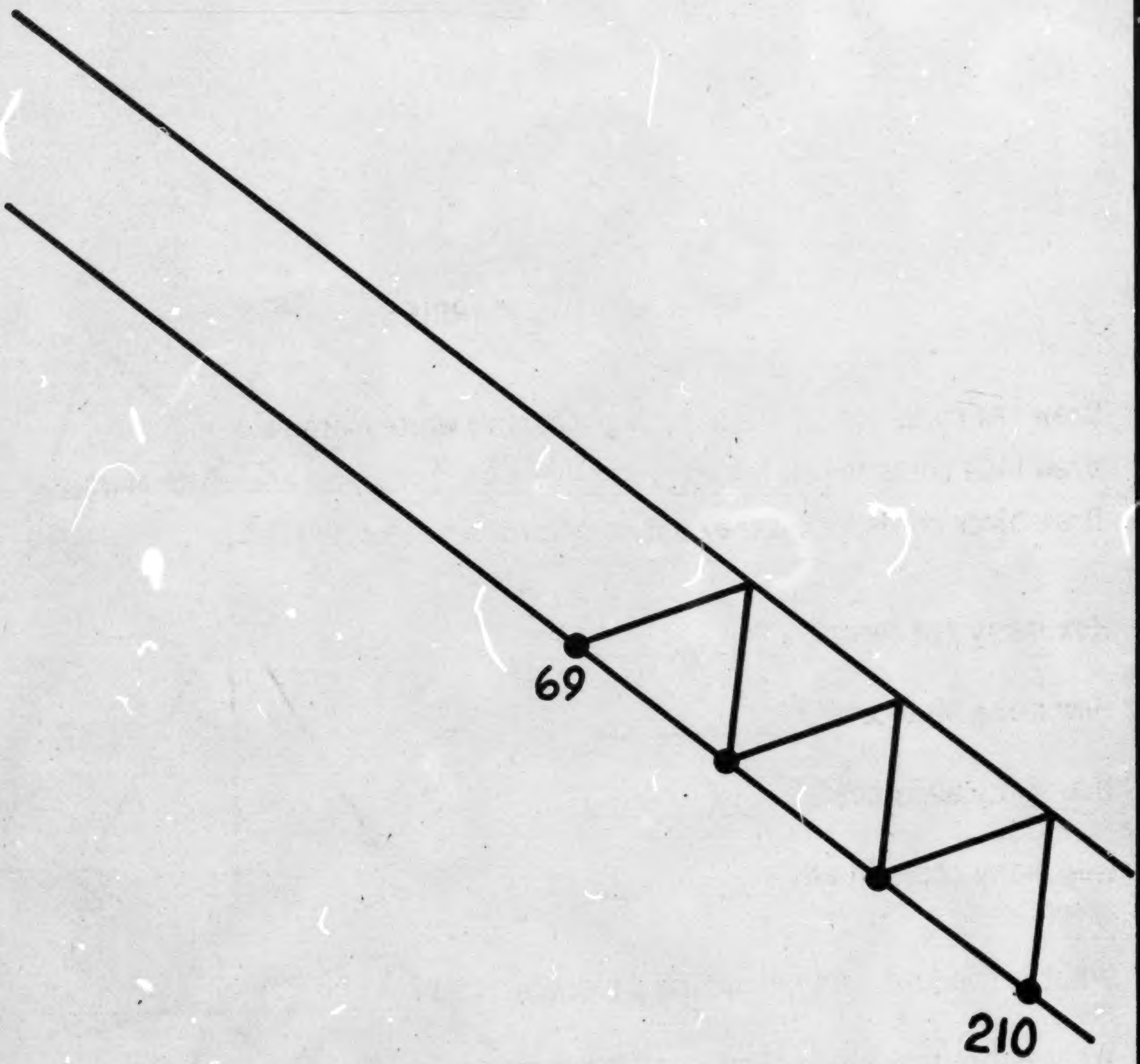




G 22 ****

Name ______________________________

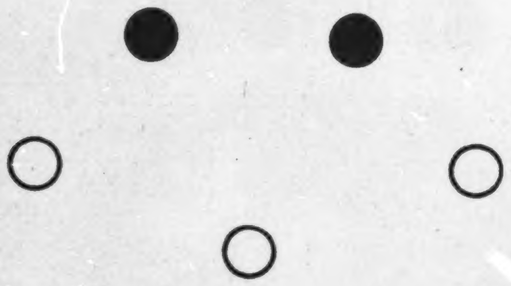
Label the dots. Use your translator to place correctly a dot for ⁻72.



Name ____________________

P 01 *

Here are two black and three white marbles.



PICK ANY TWO MARBLES

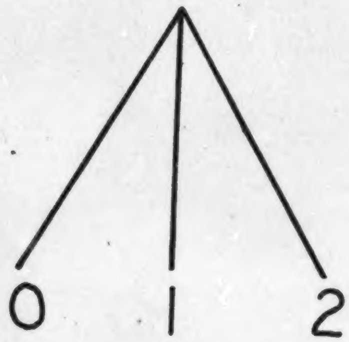
Draw red cords for all the ways of getting two white marbles.
Draw blue cords for all the ways of getting one black and one white marble.
Draw black cords for all the ways of getting two black marbles.

How many red cords? ________

How many blue cords? ________

How many black cords? ________

How many cords in all? ________



What is the probability of getting 0 black marbles? ______________

What is the probability of getting 1 black marble? ______________

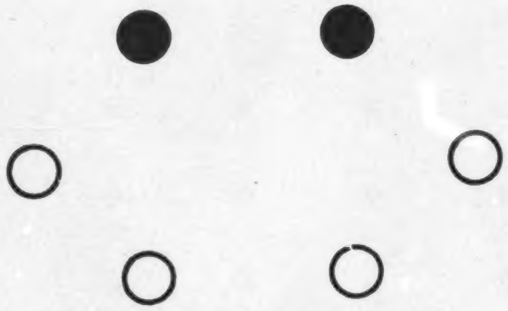
What is the probability of getting 2 black marbles? ______________

Write these probabilities in the tree picture.

P 02 **

Name ____________________

Here are two black and four white marbles.



PICK ANY TWO MARBLES

Draw red cords for all the ways of getting two white marbles.

Draw blue cords for all the ways of getting one black and one white marble.

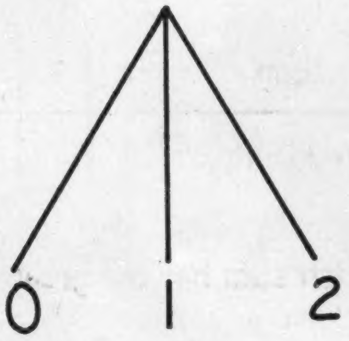
Draw black cords for all the ways of getting two black marbles.

How many red cords? __________

How many blue cords? __________

How many black cords? __________

How many cords in all? __________



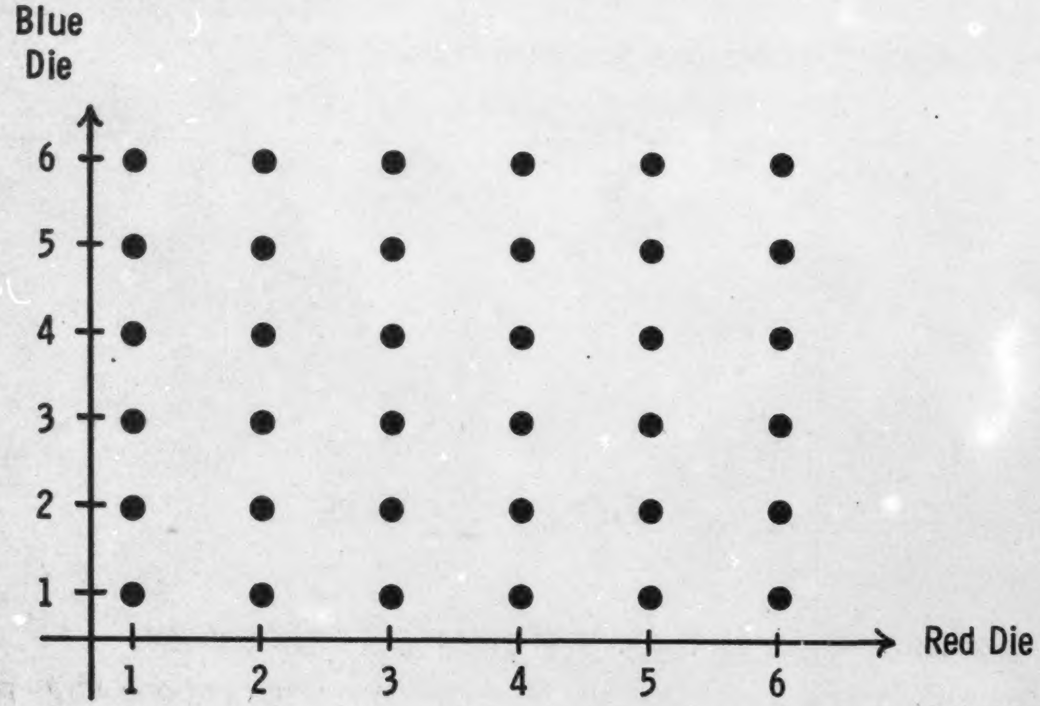
What is the probability of getting 0 black marbles? ____________________

What is the probability of getting 1 black marble? ____________________

What is the probability of getting 2 black marbles? ____________________

Write these probabilities in the tree picture.

Name ____________________ P 03 *



Complete the table.

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability											

Which sum has the greatest probability? ____________

What is the probability that the sum is less than 4? ____________

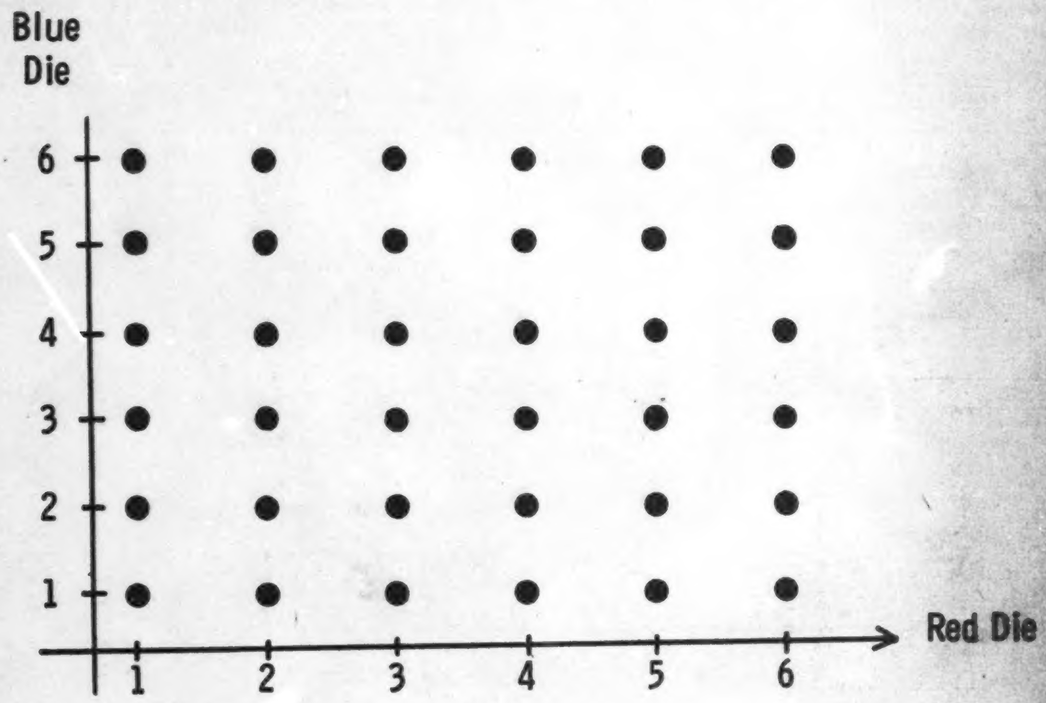
What is the probability that the sum is at most 4? ____________

What is the probability that the sum is greater than 9? ____________

What is the probability that the sum is at least 9? ____________

Name ______________________

P04 **



Refer to the Table on P03

What is the probability that the sum is even? ____________

What is the probability that the sum is odd? ____________

What is the probability that the sum is prime? ____________

What is the probability that the sum is a divisor of 12? ____________

Draw a string around the outcomes for which the same number is on both the blue and red dice.

What is the probability that the same number is on both dice? ________

Draw a string around the outcomes for which the number on the red die is greater than the number on the blue die.

What is the probability that the number on the red die is greater than the number on the blue die? ________